Integrability and the AdS/CFT Correspondence

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1 Introduction

In this dissertation, we explore the possibility of testing the AdS/CFT correspondence by examining strings with large angular momenta in the S^5 component of $AdS_5 \times S^5$ and their dual gauge theory operators, whose one loop, planar spectrum can be derived by diagonalizing the Hamiltonians of spin chains, which are known to be integrable systems. The aim is to give a clear and comprehensive review of the development which started with the discovery of the AdS/CFT correspondence and resulted in the emergence of integrable systems in the study of large R-charge string states and their corresponding gauge theory operators.

The AdS/CFT correspondence is one of the most significant advances in theoretical high energy physics in the last twenty years. The papers that first introduced the fundamental ideas by Maldacena [49], Witten [70] as well as Gubser, Klebanov and Polyakov [38] are among the most cited publications today. This shows what a dramatic influence the correspondence had and how much advances have been made, which are built upon the correspondence. It roughly states that some gravity theories are equivalent to quantum field theories which means that the two theories have the same observables, states, correlation functions and dynamics. This is an astonishing proposal since quantum field theories do not include gravity. In general, the mentioned gravity theories are string theories. String theory was first proposed in the late 1960s as a theory for strong interactions [40]. Even though this approach was abandoned, string theory soon became the most promising candidate for a grand unified theory since, for example, string spectra quite naturally include a graviton and fermions. Even though it remains unclear if string theory will one day be able to provide such a unified theory, great advances have been made such as the discovery of branes and their role of introducing gauge theories into string theory. At the same time as string theory seemed to be more and more promising, quantum field theories proved to be able to describe particle physics in an unprecedented precise manner. Non-abelian gauge theories were successfully used to describe strong interactions. They play a crucial role in the content of the standard model, from which no deviation has yet been found experimentally.

In this setting of string theory being a grand unified theory candidate and quantum field theories explaining fundamental particle physics in such a successful way, the AdS/CFT correspondence might hold the answer to the question, which of the two

theories is more fundamental. And the answer would be: none or both. If the correspondence proved to hold more generally than we have explored it until now, it might well be that we will ultimately see that both theories are completely equivalent and only two ways of describing the same underlying structure. Moreover, through the AdS/CFT correspondence string theory can be linked to QCD, the theory of strong interactions. Perhaps, the original idea of string theory as a theory of strong interactions might surprisingly not be as wrong as it seemed.

However, today we are far away from the claim that gauge theories and string theories are equivalent. Nevertheless, as mentioned above, the AdS/CFT correspondence has opened the door for many advances in different directions from defining quantum gravity in terms of gauge theory to phenomenological applications of describing strongly coupled field theories as gravity theories. Maldacena's original idea was to describe a stack of D-branes in a certain limit from two different viewpoints. Thus, he found the first and most restrictive version of the duality which states that planar $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions, a maximally supersymmetric $SU(N)$ gauge theory, is dual to type IIB supergravity in $AdS_5 \times S^5$. Today, we are convinced that the duality holds in a more general context. A stronger form of the correspondence drops the condition that the gauge theory is in the planar limit, which corresponds to type IIB string theory, rather than just supergravity, on the gravity side of the duality.

The correspondence is only a conjecture. There is no formal proof and it is very difficult to test it. We would like to identify string states and the corresponding operators in the gauge theory and then compare the energies of the string states with scaling dimensions of the operators in order to confirm the duality. The reason for the difficulties, we encounter in doing so, is that AdS/CFT is a strong/weak duality, i.e. strongly coupled gauge theories correspond to weakly coupled gravity theories and conversely. Therefore, if one side of the correspondence is accessible with perturbative tools, then the other side is not. One way to circumvent this issue is to consider BPS states, which are protected from quantum corrections by supersymmetry. Therefore, the scaling dimensions of BPS operators are not corrected in the strongly coupled case and we can compare them to the energies of BPS states in the string theory. For non-BPS states, the issue is a lot more difficult.

In 2002, Berenstein, Maldacena and Nastase found a way to circumvent this issue [16]. They found that $\mathcal{N} = 4$ SYM and type IIB string theory have an overlapping perturbative regime, in which results of calculations on both sides of the duality can be compared. Their ansatz was to find the classical solution of the string σ-model of a pointlike string stationary in AdS_5 and moving on a great circle in S^5. Then, they considered the expansion of $AdS_5 \times S^5$ string theory close to that geodesic. In doing so, they established the so-called plane-wave string/gauge theory duality by showing

that operators with high R-charges on the gauge theory side of the AdS/CFT duality correspond to strings in a plane-wave background, the Penrose limit of $AdS_5 \times S^5$. Thanks to the work of Metsaev and Tseytlin we know how to quantize string theory on a plane-wave background [51, 52]. Therefore, the string side of the duality is under control as we know all states and their energies. On the gauge theory side, the Penrose limit corresponds to a new kind of scaling limit, the so-called BMN limit. In this limit, new quantum loop and genus counting parameters arise. The new parameter that enumerates quantum loops, λ', is small on both sides of the duality at the same time. Therefore, it equips us with the possibility to perform perturbative calculations on both sides of the duality and compare the results. This offers a powerful test of the correspondence. Furthermore, the new genus counting parameter g_2 remains tuneable in the BMN limit. Therefore, we can test the duality beyond the planar limit.

Inspired by the success of Berenstein, Maldacena and Nastase, large R-charge string states and the corresponding operators were intensively studied subsequently. Berenstein, Maldacena and Nastase had restricted their treatment to pointlike strings moving on a great circle of the S^5 component of $AdS_5 \times S^5$. Gubser, Klebanov and Polyakov considered both strings rotating in AdS_5 and strings spinning in S^5 [37] and found such classical solutions of the string σ-model. These solutions made it obvious, that the ansatz of finding classical string solutions with basic global charges made it easier to find the gauge theory equivalents since these are just the states carrying the same global charges on the other side of the duality. Gubser, Klebanov and Polyakov generalised the BMN string to extended, closed, folded strings stationary in AdS_5 but rotating in one plane of the S^5. This was further generalised by Frolov and Tseytlin who, among other classical solutions, derived the solutions of folded and circular strings rotating in two orthogonal planes of the S^5 component [29, 31].

The connection of these solutions to spin chains was first discovered by Beisert, Kristjansen and Staudacher [7, 6]. They realised that the problem of finding the one loop, planar scaling dimensions of the operators corresponding to the solutions of Frolov and Tseytlin is equivalent to diagonalizing the Hamiltonian of an isotropic $s = 1/2$ spin chain. In the BMN limit, when the combined angular momentum diverges, $J_1 + J_2 \to \infty$, the spin chain goes to the thermodynamic limit where an infinite number of spins is excited. Solving this Hamiltonian eigenvalue problem yields expressions equal to the energies in the string solutions derived by Frolov and Tseytlin. The discovery, that the spin chain model is an integrable theory enormously simplifies finding the energy spectrum and therefore it helps to compare the spin chain to the corresponding string theory spectrum.

The described succession of developments is roughly the path we are taking in

this dissertation, from Maldacena's original argument to the emergence of integrability. In chapter 2, we explain the basic aspects of the AdS/CFT correspondence. The whole chapter is designed as a preparation for describing Maldacena's decoupling argument in section 2.5, the last section of that chapter. First, we study heuristic motivations for the gauge/string duality in general and the AdS/CFT correspondence in particular. We discover, that there are several clues that hint at the existence of a close relation between gauge and string theories. The most prominent of this clues is t'Hooft's $1/N$ expansion of non-abelian gauge theories with a large number of colours N, which we examine in detail. Other hints, that we explain, are the twofold interpretation of the interaction of a very light with a heavy object. The interactions can either be solved explicity or the heavy object can be treated as a deformation of the background, as it is itself uneffected by the light object. A third hint at the AdS/CFT correspondence is the open/closed string duality. After these heuristic topics, we guide through the necessary background to have enough tools at hand to understand the AdS/CFT duality and the two theories it relates. In section 2.2, we study $\mathcal{N} = 4$ Super Yang-Mills in four dimensions. We are particularly interested in the fact that it is a conformal field theory and in what conformal symmetry implies. Furthermore, we focus on the dilatation operator and scaling dimensions of fields as these give us the opportunity to directly compare gauge theory quantities with the dual expressions in string theory. In section 2.3, we give an overview of some necessary superstring theory basics. In particular, we present how gauge theories arise in string theory as the theory of open strings ending on D-branes. This insight is key to the understanding of the AdS/CFT correspondence and it is central Maldacena's decoupling argument. In the last section before we get to this decoupling argument, section 2.4, we will introduce anti-de Sitter space AdS_{d+1} and examine several coordinate representations as well as its boundary and symmetries. Moreover, we show that anti-de Sitter space arises as the near horizon limit in the 3-brane supergravity solution. Along with the emergence of gauge theories on D-branes, this is the most important insight that leads to the AdS/CFT duality. Finally, we arrive at Maldacena's decoupling argument in section 2.5. We explain in detail how the twofold interpretation of a stack of D3-branes in type IIB string theory in a certain limit lead Maldacena to his famous conjecture. Moreover, we will discuss several versions of the duality which claim the correspondence to hold under more or less restrictive conditions.

In chapter 3, we make first contact with states and operators carrying large charges. We discuss the discovery of Berenstein, Maldacena and Nastase, which we mentioned above, that superstring theory expanded around the geodesic of a pointlike string moving on a great circle in S^5 simplifies significantly and that the resulting string states and the corresponding gauge theory operators open up pos-

sibilities to test the AdS/CFT correspondence beyond the planar limit. In section 3.1, we derive that a pointlike string, that is stationary in the AdS_5 component of $AdS_5 \times S^5$, but moves on a great circle in the S^5 component, effectively sees a plane-wave background, the Penrose limit of $AdS_5 \times S^5$. In section 3.2, we derive which string states are projected out when we perform this limit. We translate the Penrose limit to the dual gauge theory in section 3.3 since we have to take the same limit on both sides of the duality to discuss the implications in the context of the AdS/CFT correspondence. In section 3.4, we quantize type IIB string theory in a plane-wave background following the work of Metsaev and Tseytlin. For simplicity, we will focus on the bosonic sector of the theory but, apart from this simplification, explain the procedure in detail. We conclude that we have full control of the string spectrum and, therefore, try to identify the dual gauge theory operators and find their scaling dimensions in section 3.5. This proves to be extremely fruitful and equips us with a tool to test the duality. In the last section of the chapter, section 3.6, we explore an example of how the BMN limit helps us to go beyond the planar sector of the correspondence.

In chapter 4, we explore the analysis of large R-charge classical string solutions as a more general tool to test the correspondence. In section 4.1, we first study the classical string σ-model solutions of extended strings rotating in either the AdS_5 or the S^5 component, which were found by Gubser, Klebanov and Polyakov. We see how the examination of classical solutions with large charges simplifies the search for the dual gauge theory operators. In section 4.2, we discuss more general classical solutions of strings spinning in S^5, which are due to Frolov and Tseytlin. These solutions describe extended strings that are circular or folded and which can rotate in more than just one plane of the S^5. We study these solutions in much detail since we want to examine the corresponding gauge theory operators using spin chains in the following chapter and then compare the results.

Finally, in chapter 5 we explore how the gauge theory equivalents of the string theory solutions, we derived in the chapters 3 and 4, can be translated to spin chains. Therefore, the quest of finding the scaling dimensions of gauge theory operators translates to solving a spin chain Hamiltonian eigenvalue problem. In section 5.1, we derive that the gauge theory operators corresponding to the solutions of strings rotating in two orthogonal planes of S^5 can be represented by $s = 1/2$ spin chains. Furthermore, we show that the one-loop dilatation operator acting on those gauge theory states reduces to the Hamiltonian of the $\text{XXX}_{1/2}$ Heisenberg spin chain in the planar limit. The Heisenberg spin chain is a known integrable theory. We will use integrability to derive eigenstates of the Hamiltonian with the help of the Bethe ansatz in section 5.2. Finally, in section 5.3, we will study the thermodynamic limit of the spin chain where an infinite number of spins is excited in the limit of a chain

of infinite length. This thermodynamic limit corresponds to the string solutions of Frolov and Tseytlin that we examined in the previous chapter. We will outline the derivation of the spectrum of the spin chain using the thermodynamic Bethe ansatz and compare it to the string theory results.

2 Introduction to the gauge/string duality

The gauge/string duality relates two seemingly unrelated theories. It was introduced by t'Hooft in the 1970s who showed that non-abelian gauge theories in a certain limit admits a perturbative formulation equivalent to a string theory [40]. Roughly, the duality states that some gauge theories are equivalent to (quantum) gravity theories, meaning that they have the same observables, states, correlation functions and dynamics. The AdS/CFT correspondence [49, 38, 70] is the first precisely formulated example of the gauge/string duality and thus substantiated the proposal by t'Hooft. The correspondence was famously proposed by Maldacena in 1997 [49] who conjectured the equivalence of $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory in $\mathbb{R}^{1,3}$ with gauge group $SU(N)$ in the planar limit and type IIB supergravity in $AdS_5 \times S^5$ in the classical limit.

The AdS/CFT correspondence is a realization of the holographic principle [41, 67]. This principle states that the dynamics of the degrees of freedom of a theory are completely determined by or equivalent to the dynamics of degrees of freedom on its boundary. The holographic principle was inspired by the result of Hawking and Beckenstein [66], that the entropy of a black hole is proportional to the area of its event horizon. This means that all information about a black hole is stored on its boundary. The AdS/CFT correspondence is a realization of this principle since the $\mathcal{N} = 4$ SYM in $\mathbb{R}^{1,3}$ can be thought of being located on the boundary of AdS_5. Both phenomena resemble a hologram which stores the information about a three dimensional image on a two dimensional surface.

The AdS/CFT correspondence is a strong/weak duality. Strongly coupled gauge theories are equivalent to weakly coupled gravity theories and conversely. Today, the duality is widely believed to hold more generally than stated above and to connect type IIB string theory in $AdS_5 \times S^5$ to $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ not necessarily in the planar limit. It is a stunning proposal, because the conjecture links a theory with gravity to one with no gravity at all. There are attempts of generalising the correspondence in order to use it to describe non-supersymmetric, non-conformal gauge theories. For example, there is interest in using the duality to describe the non-perturbative regime of QCD since it might be accessible through a

weakly coupled gravity theory by means of the correspondence. In this dissertation, however, we are more interested in the formal aspects of the correspondence. We will explore these after an introduction to the duality in this chapter.

Section 2.1 gives a heuristic introduction trying to convey an intuitive understanding of the gauge/gravity duality. This includes the concepts that lead Maldacena to his remarkable conjecture in 1997 [49]. We will see that t'Hooft claimed that gauge theories have a dual description in terms of strings as early as in the 1970s [40]. We describe how the correspondence arises from formulating a physical problem in two different ways and then comparing structures on both sides.

In section 2.2, we start to introduce the necessary tools to give a more precise description of the duality later. First, we describe $\mathcal{N} = 4$ SYM. We pay special attention to the fact that it is a conformal field theory (CFT). We explore the conclusions one can draw from the conformal symmetry with respect to operators and correlation functions. The dilatation operator and its action on fields will be of special interest. We derive the bosonic subgroup of the symmetry group of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ to be $SO(2,4) \times SO(6)$.

In section 2.3, we describe how D-branes arise in a string theory. This leads to the result that a stack of N D-branes hosts a $U(N)$ gauge theory. This is a crucial concept for the motivation of the AdS/CFT correspondence.

In section 2.4, we define the Anti-de Sitter (AdS) space and introduce different coordinate descriptions. We derive that the boundary of AdS_5 is conformally equivalent to $\mathbb{R}^{1,3}$. Consequently, we can think of $\mathcal{N} = 4$ SYM being located on the boundary of AdS_5. Moreover, we show that the symmetry group of $AdS_5 \times S^5$ is $SO(2,4) \times SO(6)$. It is therefore equivalent to the bosonic subgroup of the gauge theory on the other side of the duality. At last, we introduce the concept of p-branes as solitonic supergravity solutions in $AdS_5 \times S^5$ extended in p spatial dimensions.

In the last section of the chapter, section 2.5, we follow Maldacena's line of thought in his derivation of the AdS/CFT correspondence. We show that he derived the duality by considering a decoupling limit in two dual descriptions of the same physical situation. By comparing the structures that arise in both points of view he proposed the correspondence, which remains, however, unproven. In the end, we introduce different versions or formulations of the duality, which claim the equivalence to be true under more or less restrictive assumptions.

2.1 Heuristic motivation of the gauge/string duality

Before we derive a precise, quantitative description of the correspondence, we try to motivate it from a more heuristic point of view. Even though the gauge/string duality seems very counterintuitive at first glance – it connects a theory with gravity

to one without in a different number of dimensions – there are some observations which hint at the correspondence. We can even draw some quantitative conclusions from the heuristic viewpoints we will describe in this section.

2.1.1 T'Hooft's $1/N$ expansion of non-abelian gauge theories

The claim that string theory is a dual description of a non-abelian gauge theory with a large number of colours lies in the centre of the gauge/string duality. We can get an intuitive grasp of that relation by studying t'Hooft's insight [40] that Feynman graphs of an $SU(N)$ gauge theory can be classified according to their topology and that planar graphs dominate in the large N limit. Therefore, large N theories are simpler than real world gauge theories such as QCD since non-planar Feynman diagrams do not contribute to correlators. As we will see, the powers of λ/N count the genus of the Feynman diagram while the t'Hooft coupling $\lambda = g_{YM}^2 N$ enumerates quantum loops in the expansion of correlation functions with g_{YM} being the gauge theory Yang-Mills coupling. Instead of λ/N as genus counting parameter, one often reads that powers of $1/N$ enumerate the genus. In the planar limit, which is defined by $N \to \infty$, $g_{YM} \to 0$ such that λ is constant, this is equivalent since λ is just a constant factor. The expansion in λ/N resembles the genus expansion in string theory. This is a hint of a hidden relation to string theory which will be formulated very precisely in the AdS/CFT correspondence.

For simplicity, let us consider a zero dimensional $SU(N)$ gauge theory, a so-called matrix model of hermitian $N \times N$ matrices M (see also [33]). Let M transform as $M \to UMU^{-1}$ in the adjoint representation of $SU(N)$. Then the following purely Gaussian action

$$S[M] = -\frac{1}{g_{YM}^2}\operatorname{Tr} M^2 \tag{2.1}$$

is gauge invariant. Expectation values of operators $\mathcal{O}_i$ (assume operators are scalar for simplicity) are given by

$$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n \rangle = \frac{\int \mathcal{D}M \, \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n \, e^{-S[M]}}{\int \mathcal{D}M \, e^{-S[M]}}$$
$$= \frac{1}{Z[0]} \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \cdots \frac{\delta}{\delta J_n} Z[J_i]|_{J=0}$$

with the path integral measure $\mathcal{D}M$ and the generating functional of correlation functions $Z[J_i]$,

$$\mathcal{D}M = \prod_{i=0}^{N} dM_{ii} \prod_{i<j} d(\operatorname{Re} M_{ij}) \prod_{i<j} d(\operatorname{Im} M_{ij}), \tag{2.2}$$

$$Z[J_i] = \int \mathcal{D}M \, e^{-S[M] + J_i \mathcal{O}_i}. \tag{2.3}$$

In the case of a Gaussian action the generating functional is

$$\frac{Z[J]}{Z[0]} = \frac{\int \mathcal{D}x e^{-\frac{1}{2}x^T A x + J^T x}}{\int \mathcal{D}x \, e^{-\frac{1}{2}x^T A x}} = e^{\frac{1}{2}J^T A^{-1} J} \tag{2.4}$$

where $x = (x_1, \ldots, x_d), \mathcal{D}x = dx_1 \ldots dx_d, J = (J_1, \ldots, J_d)$ and A a symmetric $d \times d$ matrix. Then, the correlation functions are given by

$$\langle x_a x_b \rangle = A_{ab}^{-1}, \tag{2.5}$$

$$\langle x_{a_1} x_{a_2} \ldots x_{2n+1} \rangle = 0, \tag{2.6}$$

$$\langle x_{a_1} x_{a_2} \ldots x_{2n} \rangle = \sum_{\sigma \in S_{2n}} \prod_{i=1}^{n} \langle x_{a_{\sigma(2i-1)}} x_{a_{\sigma(2i)}} \rangle. \tag{2.7}$$

Equation (2.7) is Wick's theorem, which can be used to simplify n point correlation functions and trace them back to two point correlators. Next, we derive the propagator of the matrix model. First, we need to write the action in a form such that we can read off the matrix A from equation (2.4):

$$\begin{aligned}
\frac{1}{2g_{YM}^2} \operatorname{Tr} M^2 &= \frac{1}{2g_{YM}^2} \left(\sum_i M_{ii}^2 + 2 \sum_{i<j} (\operatorname{Re} M_{ij})^2 + 2 \sum_{i<j} (\operatorname{Im} M_{ij})^2 \right) \\
&= \frac{1}{2g_{YM}^2} \sum_{ijkl} M_{ij} \delta_{il} \delta_{jk} M_{kl} \\
&= \frac{1}{2} \sum_{ijkl} M_{ij} A_{ijkl} M_{kl}.
\end{aligned} \tag{2.8}$$

The last equal sign defines the Matrix A_{ijkl} with two double indices $\{ij\}, \{kl\}$ to be

$$A_{ijkl} = \frac{1}{g_{YM}^2} \delta_{il} \delta_{jk}. \tag{2.9}$$

We can now extract the propagator by inverting this matrix using equation (2.5). Since

$$\sum_{kl} \left(\underbrace{\frac{1}{g_{YM}^2} \delta_{il} \delta_{jk}}_{=A_{ijkl}} \right) \left(g_{YM}^2 \delta_{kn} \delta_{lm} \right) = \delta_{im} \delta_{jn} \tag{2.10}$$

$$\Rightarrow A_{klmn}^{-1} = g_{YM}^2 \delta_{kn} \delta_{lm} \tag{2.11}$$

we can conclude that the propagator is given by

$$\langle M_{ij} M_{kl} \rangle = g_{YM}^2 \delta_{il} \delta_{jk}, \tag{2.12}$$

which we can represent diagrammatically in the so-called double-line notation as

$$\langle M_{ij} M_{kl} \rangle = \quad \begin{array}{c} i \longrightarrow l \\ j \longleftarrow k \end{array} \quad . \tag{2.13}$$

Next, we calculate some correlators to see that the resulting Feynman diagrams do in fact organise in the fashion mentioned above. They are ordered according to their topology with planar graphs dominating in the limit of a large number of colours N with the t'Hooft coupling kept constant. As an example, let us consider the one point function of the operator $\mathrm{Tr}\, M^4 = \sum_{ijkl} M_{ij} M_{jk} M_{kl} M_{li}$. We simplify the correlator using Wick's theorem (2.7):

$$\begin{aligned}
\langle \mathrm{Tr}\, M^4 \rangle &= \left\langle \sum_{ijkl} M_{ij} M_{jk} M_{kl} M_{li} \right\rangle \\
&= 2 \sum_{ijkl} \langle M_{ij} M_{jk} \rangle \langle M_{kl} M_{li} \rangle + \sum_{ijkl} \langle M_{ij} M_{kl} \rangle \langle M_{jk} M_{li} \rangle \\
&\overset{(2.12)}{=} 2 g_{YM}^4 \sum_{ijkl} \delta_{ik} \delta_{jj} \delta_{ki} \delta_{ll} + g_{YM}^4 \sum_{ijkl} \delta_{il} \delta_{jk} \delta_{ji} \delta_{kl} \\
&= 2 g_{YM}^4 N^3 + g_{YM}^4 N = 2\lambda^2 N + \frac{\lambda^2}{N} \\
&= 2\lambda^3 \left(\frac{N}{\lambda} \right) + \lambda \left(\frac{\lambda}{N} \right) .
\end{aligned} \tag{2.14}$$

From this result, we can conclude that there are two contributing diagrams. The first has three quantum loops while the second one has only one quantum loop. The genus of the first diagram is smaller than the genus of the second one as it contributes with a higher power of λ/N. We will see a more formal statement that explains why λ/N is the parameter counting the genus of a graph later in equation (2.21).

Let us confirm that our expectations about the number of quantum loops and the genus are true. Diagrammatically, we get

$$\langle \mathrm{Tr}\, M^4 \rangle = \left\langle \sum_{ijkl} M_{ij} M_{jk} M_{kl} M_{li} \right\rangle = \left\langle \sum_{ijkl} \right\rangle$$

$$= 2 \times + . \tag{2.15}$$

The Feynman rules of the matrix model we consider do not include vertices. So,
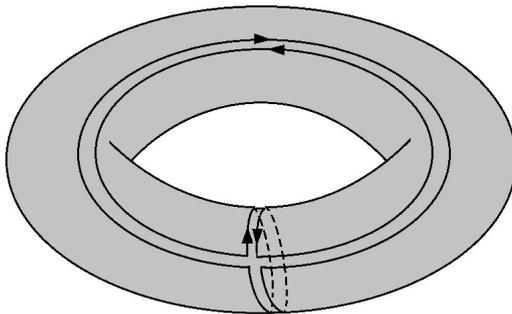
Figure 2.1: Non-planar graph from equation (2.15) drawn on a torus without cross-
ing lines.

we can only join lines with propagators and there are only distinct ways to do this.
We call the first of the two graphs planar since it does not include any crossing
of lines. In contrary, the second graph is non-planar. We see, that the planar
graph is proportional to N/λ while the non-planar one is proportional to λ/N.
Therefore, in the limit of large N and λ constant, the planar graph dominates the
expression for the correlation function. We can relate the form of the diagrams to
string worldsheets of different topologies. String worldsheets are two dimensional
surfaces. If we only consider worldsheets without boundaries, they only differ in
one topological feature which is the number of holes called the genus of the surface.
A closed two dimensional surface without holes is topologically equivalent to the
2-sphere S^2. If the surface has one hole, it is equivalent to the 2-torus T^2, etc.
In this example, we see that we can order Feynman diagrams of an $SU(N)$ gauge
theory in the same way. While the first graph can be drawn on an S^2 sphere without
crossing lines we need a torus T^2 in order to draw the second graph without crossing
lines, as can be seen in figure 2.1. If we allow vertices in the gauge theory, we will
get more diagrams requiring surfaces of higher genus in order to draw them without
crossing lines. For each diagram, there is a surface of minimal genus that allows the
diagram to be drawn on it without crossing. Therefore, we can order the diagrams
into classes according to these surfaces.

Let us consider a more complicated example, which shows that the statement
about the ordering of the graphs according to their topologies remains true when
we add interactions to the action,

$$S = -\frac{1}{2g_{YM}^2}\text{Tr}\left(M^2 + \frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \ldots\right) \tag{2.16}$$

where the t_i, $i = 3, 4, \ldots$, are coupling constants. Let us consider the connected

correlator

$$\left\langle\; :\frac{1}{g_{YM}^2}\mathrm{Tr}\,M^3:\; :\frac{1}{g_{YM}^2}\mathrm{Tr}\,M^3:\; \right\rangle_c = \left\langle\; \right\rangle_c$$

$$= \quad + \quad + \quad + \ldots (\text{ planar})$$

$$+ \quad + \ldots (\text{ one crossing})$$

$$+ \ldots (\text{ more crossings})$$

$$\sim \quad + \quad + \quad + \ldots. \qquad (2.17)$$

Again, we see that the diagrams can be organised according to their topologies. Of course, the diagrams depend on the coupling constants $t_i, i > 2$. So the matching with two dimensional Riemann surfaces is not an exact equality unless we specify precisely what kind of string theory we are considering. However, it is a qualitative statement that large N gauge theories can be expanded in a perturbation series that is resembling a perturbative genus expansion in string theory. We just cannot specify precisely the kind of string theory we obtain in this context.

After these two examples, let us make the statement about the topological expansion more precise. We follow the line of thought in [57] and explain why $(\lambda/N)^2$ enumerates the genus of a given diagram. This is important because it explains why the expansion in the genus becomes a valid perturbative expansion in the large N limit. Furthermore, we see why λ is the parameter of the perturbative quantum expansion. Each Feynman diagram in the gauge theory comes with a factor of

$$\left(g_{YM}^2\right)^{P-V} N^L \qquad (2.18)$$

where P is the number of propagators, V the number of vertices and L the number of closed loops. The factors of g_{YM}^2 come from the overall factor of $1/g_{YM}^2$ in the action. Propagators contribute with the opposite power due to the matrix inversion we performed earlier in equation (2.12). A factor of N comes from each closed loop because of the trace over colour indices.

Now, we use the result (due to Euler) that in a polyhedron the number of edges E, the number of vertices V, the number of faces F and the genus g obey

$$V - E + F = 2 - 2g. \tag{2.19}$$

In the Feynman diagrams the edges correspond to propagators and the faces to closed loops, i.e. $E = P$, $F = L$. Therefore, we can rewrite equation (2.18) as

$$\begin{aligned}
\left(g_{YM}^2\right)^{E-V} N^F &= \left(g_{YM}^2\right)^{E-V} N^{E-V} N^{V-E+F} \\
&= \left(g_{YM}^2 N\right)^{F+2g-2} N^{2-2g} \\
&= \lambda^{F+2g-2} N^{2-2g} \\
&= \lambda^F \left(\frac{\lambda}{N}\right)^{2g-2}.
\end{aligned} \tag{2.20}$$

Consequently, any correlator can be organised in a double expansion according to the genus and the number of quantum loops which is given by the faces:

$$\langle \ldots \rangle = \sum_{g=0}^{\infty} \left(\frac{\lambda}{N}\right)^{2g-2} \sum_{n=0}^{\infty} c_{g,n} \lambda^n. \tag{2.21}$$

The $c_{g,n}$ are numbers depending on the correlation function and the couplings $t_i, i > 2$. We see that the genus expansion is controlled by the expansion parameter λ/N while the t'Hooft coupling λ enumerates quantum loops. Expressing the correlator in this form also shows that the effective coupling in perturbation theory in the planar limit

$$N \to \infty \quad \text{and} \quad g_{YM} \to 0 \quad \text{but} \quad \lambda = g_{YM}^2 N = \text{const.} \tag{2.22}$$

is the t'Hooft coupling λ. We can conclude that in this planar limit only the diagrams with the lowest genus contribute. Furthermore, we deduce from the analogy to a string worldsheet expansion that the corresponding string coupling g_s in this expansion must be

$$g_s \propto \frac{\lambda}{N} = g_{YM}^2. \tag{2.23}$$

Here, we choose a convention where

$$4\pi g_s = g_{YM}^2. \tag{2.24}$$

Therefore, we have already identified two of the four parameters on the gauge theory and string theory side of the correspondence. The second equation relating the two will be derived later in equation (2.123). One expects the strict planar limit on the
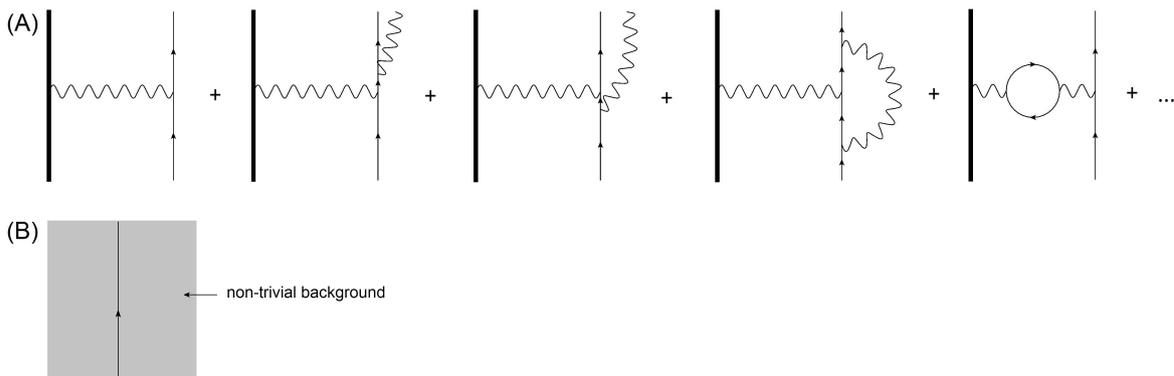
Figure 2.2: (A) Perturbative expansion of the interaction of a heavy proton with an electron. (B) The electron moving in the background deformed by the proton.

gauge theory side where

$$\frac{\lambda}{N} = 0 \qquad \Rightarrow \qquad g_s = 0 \tag{2.25}$$

to correspond to a non-interacting, free string theory.

Our restriction to a zero dimensional matrix model was just a simplification. Everything goes through analogously if we increase the dimension of spacetime [40]. If we go to a higher number of dimensions and replace $M^2 \to \partial_\mu M \partial^\mu M$, the computation of the Feynman diagrams is more complicated. However, the conclusion is still be that the Feynman diagrams describe a string theory. In particular, the same planar limit also occurs for $\mathcal{N} = 4$ SYM for large N (see [57, 23]).

2.1.2 Feynman diagrams versus non-trivial background

If we want to study the motion of an electron in the vicinity of a heavy proton, with which it interacts electromagnetically, there are two ways to treat this problem:

(A) explicit calculation of Feynman diagrams or

(B) studying the electron in a non-trivial background.

These two viewpoints are illustrated in figure 2.2. We could perform the standard QED calculation using a perturbative expansion. However, since we are not interested in the dynamics of the proton, which we assume to be unaffected by the electron, we can effectively treat it as a deformation of the trivial background. Then, we can study the motion of the electron in that background. Note that we can use the same reasoning when we study the interaction of a string with a stack of D-branes. Either we can use string perturbation theory and an expansion in different worldsheets ordered by their genus or we consider a free string propagating
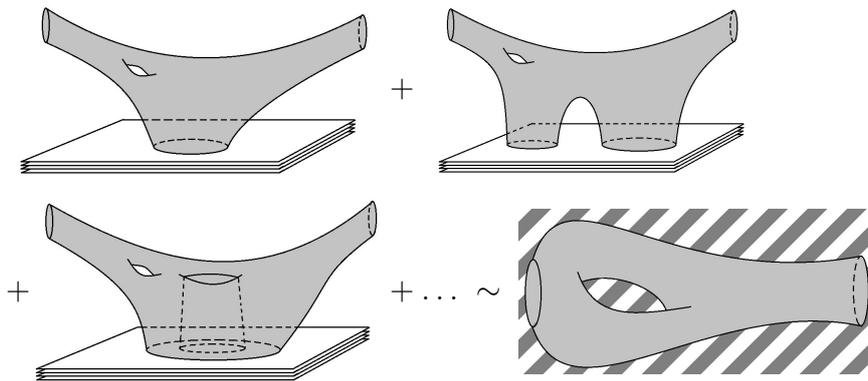
Figure 2.3: A string interacting with a stack of branes. A perturbative expansion in the string worldsheets of different topologies is equivalent to treating the branes as a deformation of the background through which a closed string propagates.

in the background deformed by the stack of D-branes, see figure 2.1.2. When we follow Maldacena's line of thoughts, which led to the AdS/CFT conjecture, later in section 2.5, we will consider a stack of branes just as we did here. The two interpretations of the stack trace back to the fact that branes are both solutions to the low energy string effective action (i.e. supergravity) and that they host a $U(N)$ gauge theory by means of the Chan-Paton mechanism, which is described in a set of published lecture notes by Polchinski [59].

2.1.3 Open/closed string duality

The above twofold interpretation can also be related to the open/closed string duality which is due to the two topologies a string can have. We know an open string coupling to a stack of branes and closed strings emitted by the brane are essentially the same. These two processes can be seen in figure 2.4. The former process leads to a gauge theory while the latter describes gravitons emitted by the brane deforming the background. Again, we see the equivalent description of the same situation treated once as a gauge theory and once as a gravity phenomenon.

2.2 $\mathcal{N} = 4$ Super Yang-Mills

$\mathcal{N} = 4$ SYM in $\mathbb{R}^{1,3}$ with gauge group $SU(N)$ is on the gauge theory side of the AdS/CFT correspondence. It is a supersymmetric, conformal quantum field theory. A comprehensive review of the subject is [23] by D'Hoker and Freedman.
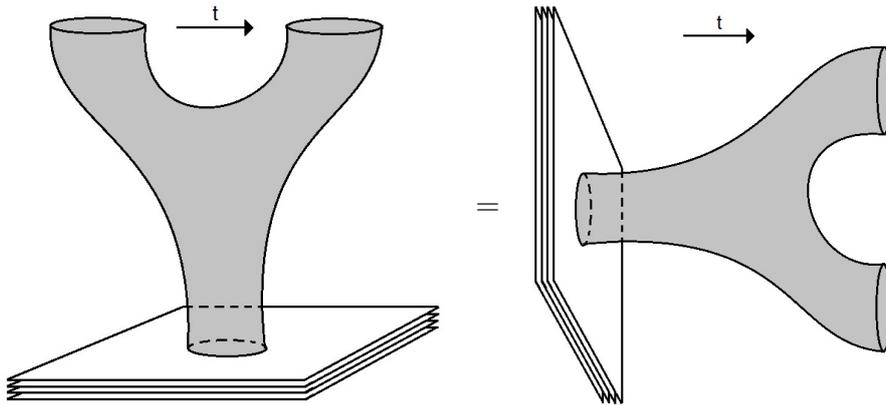
Figure 2.4: Open/closed string duality: An open string interacting with a stack of branes is equivalent to a brane emitting a pair of closed strings.

2.2.1 Dimensional reduction of 10d $\mathcal{N} = 1$ SYM

$\mathcal{N} = 4$ SYM theory in 4d can be obtained from $\mathcal{N} = 1$ SYM theory in 10d by dimensional reduction on the torus T^6. This was first done by Brink, Schwarz and Scherk [21]. The theory is the maximally supersymmetric $SU(N)$ gauge theory that does not contain gravity, i.e. no spin-2 particle.

The ten dimensional theory is defined by the action [63]

$$S_{\mathcal{N}=1,10d} = \int d^{10}x \ \mathrm{Tr} \left(\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} \Psi \Gamma^M \mathcal{D}_M \Psi \right) \tag{2.26}$$

where $(A_M)_a{}^b$, $\Psi_a{}^b$, $M \in \{0, \ldots, 9\}$, $a, b \in \{1, \ldots, N\}$ are a gauge field and a gaugino in the adjoint representation of the gauge group $SU(N)$ and the gamma matrices Γ^M obey the Clifford algebra relation $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$. The field strength and the covariant derivative are

$$F^{MN} = i[\mathcal{D}^M, \mathcal{D}^N], \tag{2.27}$$

$$\mathcal{D}_M = \partial_M - ig_{YM}[A_M, \ \cdot \ ]. \tag{2.28}$$

Toroidal compactification on T^6 breaks the Lorentz group,

$$SO(1,9) \to SO(1,3) \times SO(6). \tag{2.29}$$

We can restrict our treatment to the massless little group $SO(8)$, a subgroup of the Lorentz group, whose representations can be boosted to the full Lorentz group [42]. Upon compactification the little group breaks according to

$$SO(8) \to SO(2) \times SO(6). \tag{2.30}$$

Therefore, the fundamental vector representation and the spinor representation break according to:

$$[1000]_8 \quad \rightarrow \quad [100]_6 + (q^{-2} + q^2)[000]_6$$

$$[0001]_8 \quad \rightarrow \quad q[010]_6 + q^{-1}[001]_6$$

$$\Rightarrow \quad A_M \quad \rightarrow \quad \begin{cases} A_\mu, & \mu = 0, \dots, 3 \\ \phi_i \equiv A_i, & i = 4, \dots, 9 \end{cases}$$

$$\Psi \quad \rightarrow \quad \psi_a, \overline{\psi}^a, a = 1, \dots, 4 \tag{2.31}$$

Therefore, the compactification yields a supermultiplet of a vector A_μ, six scalars ϕ_i and two fermions in the fundamental and antifundamental of the R-symmetry group $SO(6) \cong SU(4)$.

A detailed analysis yields the action for the $\mathcal{N} = 4$ theory in 4d to be [63]:

$$S_{\mathcal{N}=4,4d} = \int d^4x \, \mathrm{Tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \mathcal{D}_\mu \phi^i \mathcal{D}^\mu \phi_i - \frac{g_{YM}^2}{4} [\phi_i, \phi_j][\phi^i, \phi^j] \right.$$
$$\left. + \overline{\psi}^a \sigma^\mu D_\mu \psi_a - \frac{ig_{YM}}{2} \sigma_i^{ab} \psi_a [\phi_i, \psi_b] - \frac{ig_{YM}}{2} \sigma_{ab}^i \overline{\psi}^a [\phi_i, \overline{\psi}^b] \right) \tag{2.32}$$

where σ^μ and σ^i are the four- and six-dimensional chiral projections of the gamma matrices Γ^M. From the point of view of this compactification we can interpret the $SO(6) \cong SU(4)$ R-symmetry of $\mathcal{N} = 4$ SYM as rotations in the compactified dimensions. For $\chi \in \{\phi_i, \psi, F_{\mu\nu}, \mathcal{D}_\mu \phi_i, \mathcal{D}_\mu \psi, \dots\}$, χ transforms covariantly in the adjoint of the gauge group. Therefore, the trace over a string of such operators is gauge invariant, e.g.

$$\mathcal{O}(x) = \mathrm{Tr} \left(\chi_1(x) \chi_2(x) \dots \chi_n(x) \right). \tag{2.33}$$

Hence, the relevant operators of the theory are single- or multi-trace gauge invariant expressions of this type.

2.2.2 $\mathcal{N} = 4$ Super Yang-Mills as conformal field theory

We know that the β-function for any non-abelian gauge theory at one loop is [36]

$$\beta(g_{YM}) = \frac{\partial g_{YM}}{\partial \log \mu} = \mu \frac{\partial g_{YM}}{\partial \mu}$$
$$= g_{YM}^3 \left(\frac{11}{3} N - \frac{1}{6} N_s N - \frac{1}{6} N_f N \right) \tag{2.34}$$

where N is the number of colours, N_s and N_f the number of scalars and (real) fermions and μ the energy scale. For $N_s = 6$ and $N_f = 16$ as in $\mathcal{N} = 4$ SYM, we get

$$\beta(g_{YM}) = 0. \tag{2.35}$$

This is even true to all loop orders [50, 64, 20]. The theory is therefore invariant under scale transformations even at the quantum level and it turns out $\mathcal{N} = 4$ SYM is a conformal field theory. A standard reference for CFTs is Ginsparg's review [32]. More details can be found in the books [60, 27].

Conformal field theories are field theories invariant under the conformal group. They have no intrinsic length or energy scale. The conformal group is defined as the set of transformations $x^\mu \to \tilde{x}^\mu(x)$, $\mu = 0, \ldots d - 1$, that leave the metric invariant up to a local scale factor:

$$\eta_{\rho\sigma} \frac{\partial \tilde{x}^\rho}{\partial x^\mu} \frac{\partial \tilde{x}^\sigma}{\partial x^\nu} = \Omega^2(x)\eta_{\mu\nu}. \tag{2.36}$$

The conformal group contains the Poincaré group and in dimension $d > 2$ it further consists of the following transformations, written in infinitesimal form:

- Dilatations: $x^\mu \to \tilde{x}^\mu(x) = x^\mu + \lambda x^\mu$, $\lambda \in \mathbb{R}^+$,
- Special conformal transformations: $\mu \to \tilde{x}^\mu(x) = x^\mu + b^\mu x^2 - 2x^\mu(b \cdot x)$, $b^\mu \in \mathbb{R}^{1,3}$.

The full proof can be found in Ginsparg's review [32]. Here, we state the crucial steps. Consider an infinitesimal conformal transformation of the form

$$\tilde{x}^\mu(x) = x^\mu + \xi^\mu(x), \qquad \Omega(x) = 1 + \omega(x) \tag{2.37}$$

where $\xi^\mu, \omega \ll 1$. Plugging into equation (2.36) yields

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2\omega(x)\eta_{\mu\nu} \tag{2.38}$$

to first order. Contracting with $\eta^{\mu\nu}$, we find

$$\omega(x) = \frac{1}{d}\partial \cdot \xi, \tag{2.39}$$

with d the number of spacetime dimensions. We therefore obtain the conformal Killing equation,

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{d}(\partial \cdot \xi)\eta_{\mu\nu}. \tag{2.40}$$

The most general solution for $d > 2$ is

$$\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu(b \cdot x) \tag{2.41}$$

where a^μ, b^μ, λ are arbitrary and $\omega_{\mu\nu}$ is antisymmetric. These parametrize translations, special conformal transformations, dilatations and Lorentz transformations, respectively.

The dimension of the conformal group in $d > 2$ dimensions is

$$\dim SO(2,d) = \underbrace{\frac{1}{2}d(d-1)}_{\text{antisym. Lorentz } \omega_{\mu\nu}} + \underbrace{d}_{\text{transl. } a^\mu} + \underbrace{1}_{\text{dilatation } \lambda} + \underbrace{d}_{\text{spec. conf. } b^\mu}$$

$$= \frac{1}{2}(d+2)(d+1). \tag{2.42}$$

Together with the generator of dilatations D and the generator of special conformal transformations K^μ the Poincaré generators $M^{\mu\nu}$ and P^μ form the conformal algebra:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho}M^{\nu\sigma} + 3 \text{ terms}),$$
$$[M^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu),$$
$$[M^{\mu\nu}, K^\rho] = i(\eta^{\mu}\rho K^\nu - \eta^{\nu\rho}K^\mu),$$
$$[M^{\mu\nu}, D] = 0,$$
$$[D, P^\mu] = -iP^\mu,$$
$$[D, K^\mu] = iK^\mu,$$
$$[P^\mu, K^\nu] = -2i(\eta^{\mu\nu}D + M^{\mu\nu}). \tag{2.43}$$

In a Lorentzian spacetime this algebra is isomorphic to $SO(2,d)$. We can see this by defining the antisymmetric matrix

$$\overline{M}^{NK} = \begin{pmatrix} M^{\mu\nu} & \overline{M}^{\mu,d} & \overline{M}^{\mu,d+1} \\ -\overline{M}^{\nu,d} & 0 & D \\ -\overline{M}^{\nu,d+1} & -D & 0 \end{pmatrix} \tag{2.44}$$

where $\mu, \nu = 0, 1, \ldots d-1$ and

$$\overline{M}^{\mu,d} = \frac{K^\mu - P^\mu}{2}, \qquad \overline{M}^{\mu,d+1} = \frac{K^\mu + P^\mu}{2}. \tag{2.45}$$

It is straightforward but tedious to check that

$$\left[\overline{M}^{NK}, \overline{M}^{LR}\right] = i\left(\eta^{NL}\overline{M}^{KR} + 3 \text{ terms}\right), \tag{2.46}$$

with $\eta_{NK} = \text{diag}(-1, 1, \ldots, 1, -1)$. Here, we present the proof that $\eta_{d+1,d+1} = -1$

and the rest can be calculated analogously:

$$
\begin{aligned}
\left[\overline{M}^{d+1,\nu}, \overline{M}^{d+1,\rho}\right] &= \frac{1}{4}[K^\nu + P^\nu, K^\rho + P^\rho] \\
&= \frac{1}{4}([K^\nu, P^\rho] + [P^\nu, K^\rho]) \\
&\overset{(2.43)}{=} \frac{1}{4}\left(2i(\eta^{\rho\nu}D + M^{\rho\nu}) - 2i(\eta^{\nu\rho}D + M^{\nu\rho})\right) \\
&= -iM^{\nu\rho} \\
&\overset{!}{=} i\Big(\eta^{d+1,d+1}\overline{M}^{\nu\rho} \underbrace{-\eta^{\nu,d+1}\overline{M}^{d+1,\rho} - \eta^{d+1,\rho}\overline{M}^{\nu,d+1}}_{=0 \text{ from looking at other equations}} + \eta^{\nu\rho}\underbrace{\overline{M}^{d+1,d+1}}_{=0}\Big)
\end{aligned}
\tag{2.47}
$$

where $\nu, \rho = 0, 1, \ldots, d-1$. Therefore,

$$
\eta^{d+1,d+1} = -1. \tag{2.48}
$$

This is just one example. Since we get the algebra structure in equation (2.46), this means that the $\overline{M}^{NK}$ obey the algebra of $SO(2,d)$. Hence, the algebra of the conformal group in d dimensions is isomorphic to $SO(2,d)$. We will later rediscover this group as the symmetry group of AdS_{d+1}-space. Naturally, it is crucial for the symmetries to match on both sides of the duality.

Since we are considering SYM theory we have to consider supersymmetry in addition. In supersymmetric field theories the spacetime symmetry group is extended by additional fermionic generators. In 4d, these are

$$
Q_\alpha^a, \overline{Q}_{\dot\alpha a} \qquad \text{where } \alpha = 1, 2, a = 1, \ldots, \mathcal{N}. \tag{2.49}
$$

They are Weyl spinors of the Lorentz group and commute with translations. Furthermore, they obey

$$
\{Q_\alpha^a, \overline{Q}_{\dot\beta b}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^a_b, \qquad \{Q_\alpha^a, Q_\beta^b\} = 2\epsilon_{\alpha\beta}Z^{ab}. \tag{2.50}
$$

Z^{ab} is antisymmetric and commutes with all elements of the algebra. It is called a central charge. For $\mathcal{N} = 1$, antisymmetry requires $Z = 0$. The supersymmetry algebra is invariant under rotations among the supersymmetry generators Q_α^a which yields the so-called R-symmetry group $SU(\mathcal{N})_R$. The supersymmetry generators have an additional global $U(1)_R$ symmetry. Both $SU(\mathcal{N})_R$ and $U(1)_R$ can be broken by quantum effects. The Haag-Lopuszanski-Sohnius Theorem [39] (a supersymmetric extension of the famous Coleman-Mandula Theorem) states that the largest possible symmetry group of a four dimensional quantum field theory is the supersymmetric extension of $SO(2,4)$, called $PSU(2,2|4)$, times an internal sym-

metry group. In that sense $\mathcal{N} = 4$ SYM is the maximally supersymmetric QFT in four dimensions. $PSU(2,2|4)$ contains $SO(2,4) \times SO(6)$ as bosonic subgroup, which corresponds to the conformal symmetry and the R-symmetry.

2.2.3 The operator/state correspondence

In conformal field theories in $\mathbb{R}^d$ there is a remarkable link between operators and states. States in the field theory are in one-to-one correspondence to operators inserted at the origin. This is very striking since operators inserted at a point are local objects, while states are obviously non-local as they carry information about field configurations. A detailed treatment can be found in the lecture notes [68]. We give an intuitive explanation and restrict ourselves to $d = 4$.

We can write the Euclidean metric (after performing a Wick rotation in the Lorentzian case) of $\mathbb{R}^4$ as

$$ds^2 = dr^2 + r^2 ds_{S^3}^2 = e^{2\tau} \underbrace{(d\tau^2 + ds_{S^3}^2)}_{\text{cylinder } \mathbb{R} \times S^3} \tag{2.51}$$

where we have substituted $r = e^\tau$. Consequently, the plane is conformally equivalent to the cylinder $\mathbb{R} \times S^3$ since the two are only related by a rescaling (see figure 2.5). A dilatation on the plane corresponds to a translation in the τ-direction on the cylinder and

$$r = 0 \leftrightarrow \tau = -\infty,$$
$$r = \infty \leftrightarrow \tau = \infty. \tag{2.52}$$

If we have a state in the CFT in $d = 4$, we can map it to the cylinder, evolve it back to $\tau = -\infty$ on the cylinder and then map back to $\mathbb{R}^4$. The state then becomes a local disturbance at $r = 0$. So there is a one-to-one correspondence between states in a CFT and operators inserted at the origin.

The fact that dilatations in $\mathbb{R}^4$ are equivalent to time translations on the cylinder motivates

$$\{\Delta\}_{\mathbb{R}^4} = \{E\}_{\mathbb{R} \times S^3}, \tag{2.53}$$

i.e. the spectrum of scaling dimensions Δ of operators is equivalent to the spectrum of energies E of states. Consequently, the dilatation operator D can be interpreted as a Hamiltonian. For this reason, we examine the dilatation operator and scaling dimensions in more detail.
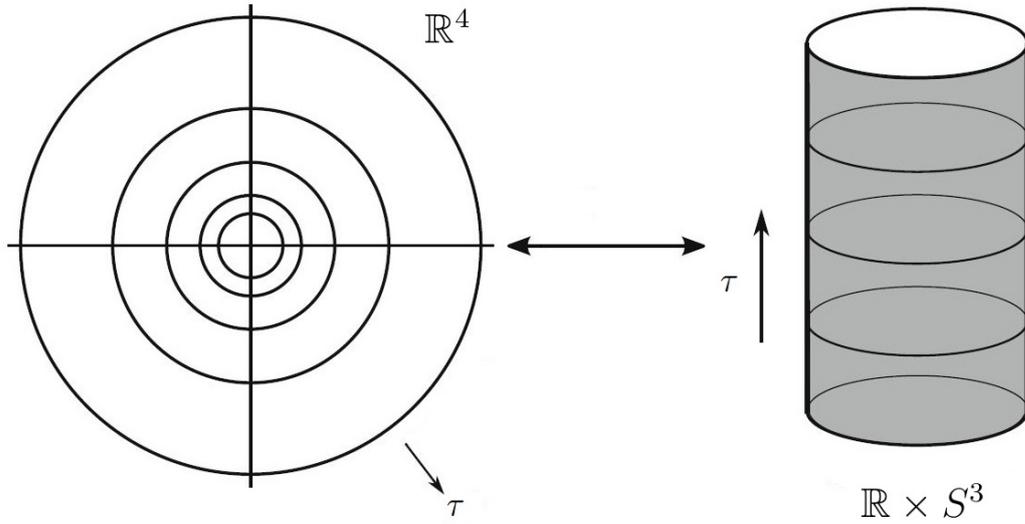
Figure 2.5: States in a conformal field theory are in one-to-one correspondence to operators at the origin. The conformal map between $\mathbb{R}^4$ and $\mathbb{R} \times S^3$ maps circles in the projection of $\mathbb{R}^4$ to slices of the cylinder. Dilatations in $\mathbb{R}^4$ correspond to time translations (i.e. translations in τ-direction) on the cylinder.

2.2.4 Dilatation operator and scaling dimensions

In addition to the Poincaré labels, states (or the equivalent operators) in a conformal field theory carry an additional label, the scaling dimension Δ, associated to the additional Cartan generator D. An operator $\phi_\Delta(x)$ has scaling dimension Δ if

$$[D, \phi_\Delta(0)] = -i\Delta\phi_\Delta. \tag{2.54}$$

This is equivalent to

$$[D, \phi_\Delta(x)] = -i\Delta\phi_\Delta(x) - x_\mu[P^\mu, \phi_\Delta(x)], \tag{2.55}$$

which can be derived using

$$\phi_\Delta(x) = e^{-iP \cdot x}\phi_\Delta(0)e^{iP \cdot x} \qquad \text{and} \qquad [D, e^{iP \cdot x}] \overset{(2.43)}{=} (P \cdot x)e^{iP \cdot x}. \tag{2.56}$$

Consequently, it is always sufficient to examine the transformation behaviour of a field at $x = 0$.

Reconsidering the conformal algebra (2.43), we see, that P^μ and K^μ can be interpreted as raising and lowering operators with respect to the dilatation operator:

$$[D, P^\mu] = -iP^\mu \quad \Rightarrow \quad [D\left[P^\mu, \phi_\Delta(0)\right]] = -[P^\mu, [\phi_\Delta(0), D]] - [\phi_\Delta(0), [D, P^\mu]]$$

$$= -i\Delta[P^\mu, \phi_\Delta(0)] + i[\phi_\Delta(0), P^\mu]$$

$$= -i(\Delta + 1)[P^\mu, \phi_\Delta(0)],$$

$$[D, K^\mu] = iK^\mu \quad \Rightarrow \quad [D, [K^\mu, \phi_\Delta(0)]] = -[K^\mu, [\phi_\Delta(0), D]] - [\phi_\Delta(0), [D, K^\mu]]$$

$$= -i\Delta[K^\mu, \phi_\Delta(0)] - i[\phi_\Delta(0), K^\mu]$$

$$= -i(\Delta - 1)[K^\mu, \phi_\Delta(0)]. \tag{2.57}$$

We now use our intuition from the harmonic oscillator. If we have a lowering operator there must be a state in each representation with minimal scaling dimension. These so-called primary operators are the operators corresponding to the lowest weight states in a representation of the conformal group. We therefore conclude that for a primary ϕ_Δ we get the commutator

$$[K^\mu, \phi_\Delta(0)] = 0. \tag{2.58}$$

We obtain the other states in the representations by acting on the primary with the momentum operator P^μ. In the language of fields this means that all states in a representation of the conformal group are given by the derivatives of a primary field. These are the so-called descendant fields. There is an infinite number of those as could be expected from the non-compactness of $PSU(2,2|4)$.

Let us consider a primary $\phi_\Delta^A(x)$ with scaling dimension Δ and an index A in the representation R of the Lorentz group. Then,

$$[M^{\mu\nu}, \phi_\Delta^A(0)] = (M_R^{\mu\nu})^A{}_B \phi_\Delta^B(0),$$

$$[D, \phi_\Delta^A(0)] = -i\Delta \phi_\Delta^A(0),$$

$$[K^\mu, \phi_\Delta^A(0)] = 0. \tag{2.59}$$

We can derive the commutators $[Q, \phi_\Delta^A(x)]$ for $Q \in \{M^{\mu\nu}, D, K^\mu\}$ using

$$\phi_\Delta^A(x) = e^{-iP\cdot x} \phi_\Delta^A(0) e^{iP\cdot x} \tag{2.60}$$

as well as the commutators (2.59). We can compute the action of $Q \in \{M^{\mu\nu}, D, K^\mu\}$ on descendant operators using that we get descendant operators by acting on primary ones with the momentum operators P^μ and that we know the commutator of P^μ with the other operators in the conformal algebra (2.43).

Under a conformal transformation $x^\mu \to \tilde{x}^\mu(x)$ primary fields transform as:

- a scalar:

$$\tilde{\phi}(\tilde{x}) = \left| \frac{\partial x}{\partial \tilde{x}} \right|^{-\Delta/d} \phi(x) \tag{2.61}$$

  where d is the spacetime dimension and Δ the scaling dimension of the operator and

- a general Lorentz representation:

$$\tilde{\phi}^A(\tilde{x}) = \left|\frac{\partial x}{\partial \tilde{x}}\right|^{-\Delta/d} L^A{}_B(R)\phi^B(x) \tag{2.62}$$

where the matrices $L^A{}_B(R)$ form a representation of the Lorentz group and $R^\mu{}_\nu = \Omega^{-1}(x)\frac{\partial \tilde{x}^\mu}{\partial x^\nu}$ is a spacetime dependent Lorentz transformation since

$$\eta_{\mu\nu}R^\mu{}_\rho R^\nu{}_\sigma = (\Omega^{-1}(x))^2\eta_{\mu\nu}\frac{\partial \tilde{x}^\mu}{\partial x^\rho}\frac{\partial \tilde{x}^\nu}{\partial x^\sigma} = \eta_{\rho\sigma}. \tag{2.63}$$

For $\tilde{x}^\mu(x) = x^\mu + \xi^\mu(x)$ with the infinitesimal vector $\xi^\mu(x)$ from equation (2.41) one can show that a primary field ϕ^A of scaling dimension Δ transforms as

$$\delta\phi_\Delta^A(x) = \underbrace{-\xi^\mu\partial_\mu\phi_\Delta^A(x)}_{\text{orbital}} + \underbrace{\frac{i}{2}\Omega_{\mu\nu}(M^{\mu\nu})^A{}_B\phi_\Delta^B(x)}_{\text{Lorentz}} \underbrace{-\Delta\omega(x)\phi_\Delta^A(x)}_{\text{scale}} \tag{2.64}$$

where

$$\Omega_{\mu\nu}(x) = \omega_{\mu\nu} - 2(x_\mu b_\nu - x_\nu b_\mu), \tag{2.65}$$

$$\omega(x) = \frac{1}{d}(\partial \cdot \xi) = \lambda - 2x \cdot b. \tag{2.66}$$

We will use these transformation rules in the next subsection to derive general expressions for correlation functions in CFTs.

2.2.5 Correlation functions in conformal field theories

Correlation functions in CFTs are covered in Blumenhagen and Plauschinn's book [60]. Conformal invariance strongly restricts the form of correlation functions in conformal field theories. They must be conformally invariant because they are observables. This means, if $x \to \tilde{x}(x)$ is a conformal transformation and $\mathcal{O}_i^{A_i}$ are operators with dimensions Δ_i and Lorentz labels A_i, then

$$\langle \tilde{\mathcal{O}}_1^{A_1}(\tilde{x}_1)\tilde{\mathcal{O}}_2^{A_2}(\tilde{x}_2)\dots\tilde{\mathcal{O}}_n^{A_n}(\tilde{x}_n)\rangle = \langle \mathcal{O}_1^{A_1}(\tilde{x}_1)\mathcal{O}_2^{A_2}(\tilde{x}_2)\dots\mathcal{O}_n^{A_n}(\tilde{x}_n)\rangle. \tag{2.67}$$

Let us consider the different n point correlators specializing to Lorentz scalar operators:

- Let $\mathcal{O}_\Delta$ be a scalar operator of scaling dimension Δ. Then the one point function is

$$\langle \mathcal{O}_\Delta(x)\rangle = \text{const.} = C \tag{2.68}$$

by translational invariance. For a scale transformation $\tilde{x}^\mu = \lambda x^\mu$, we get

$$C = \langle \mathcal{O}_\Delta(\tilde{x}) \rangle \overset{(2.67)}{=} \langle \tilde{O}_\Delta(\tilde{x}) \rangle$$

$$\overset{(2.61)}{=} \lambda^{-\Delta} \langle \mathcal{O}_\Delta(x) \rangle = \lambda^{-\Delta} C. \tag{2.69}$$

Therefore, the one point correlator is

$$\langle \mathcal{O}_{\Delta \neq 0}(x) \rangle = 0, \qquad \langle \mathcal{O}_{\Delta = 0}(0) \rangle = C. \tag{2.70}$$

- Now, consider the two point correlator of $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$ with dimensions Δ_1, Δ_2. Due to translational invariance the correlator $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \rangle$ can only be a function of the distance $|x_1 - x_2|$, i.e. $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \rangle = F(|x_1 - x_2|)$. For a dilatation, we get

$$F(|\tilde{x}_1 - \tilde{x}_2|) = \langle \mathcal{O}_1(\tilde{x}_1)\mathcal{O}_2(\tilde{x}_2) \rangle$$

$$\overset{(2.67)}{=} \langle \tilde{O}_1(\tilde{x}_1)\tilde{O}_2(\tilde{x}_2) \rangle$$

$$\overset{(2.61)}{=} \lambda^{-(\Delta_1 + \Delta_2)} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \rangle$$

$$= \lambda^{-(\Delta_1 + \Delta_2)} F(|x_1 - x_2|). \tag{2.71}$$

Therefore, the two point function has the form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \rangle = \frac{C}{|x_1 - x_2|^{\Delta_1 + \Delta_2}}. \tag{2.72}$$

The correlator gets further restricted by invariance under special conformal transformations. One can express a special conformal transformation K^μ as $K^\mu = I P^\mu I$ where I is an inversion,

$$I : \ \tilde{x}^\mu = \frac{x^\mu}{x^2}. \tag{2.73}$$

Imposing inversion invariance leads to

$$\langle \mathcal{O}_1(\tilde{x}_1)\mathcal{O}_2(\tilde{x}_2) \rangle \overset{(2.67)}{=} \langle \tilde{O}_1(\tilde{x}_1)\tilde{O}_2(\tilde{x}_2) \rangle$$

$$= \frac{1}{(\tilde{x}_1^2)^{\Delta_1}} \frac{1}{(x_2^2)^{\Delta_2}} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \rangle$$

$$\overset{(2.72)}{\Rightarrow} \frac{C}{|\tilde{x}_1 - \tilde{x}_2|^{\Delta_1 + \Delta_2}} = \frac{1}{(\tilde{x}_1^2)^{\Delta_1}} \frac{1}{(x_2^2)^{\Delta_2}} \frac{C}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} \tag{2.74}$$

Using

$$\frac{\tilde{x}_1^2 \tilde{x}_2^2}{|\tilde{x}_1 - \tilde{x}_2|^2} = \frac{1}{|x_1 - x_2|^2} \tag{2.75}$$

and plugging into equation (2.74), we obtain

$$\frac{\left(\tilde{x}_1^2\right)^{\Delta_1}\left(\tilde{x}_2^2\right)^{\Delta_2}}{|\tilde{x}_1 - \tilde{x}_2|^{\Delta_1+\Delta_2}} = \left[\frac{\tilde{x}_1^2\tilde{x}_2^2}{|\tilde{x}_1 - \tilde{x}_2|^2}\right]^{\frac{\Delta_1+\Delta_2}{2}}, \tag{2.76}$$

which can only be generally true if

$$\Delta_1 = \Delta_2. \tag{2.77}$$

So we finally get the two point correlation function,

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle = \frac{C\delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{2\Delta_1}}. \tag{2.78}$$

- Also three point functions have a similar appearance by analogous arguments. We apply translation and dilatation invariance to get

$$\langle \mathcal{O}_1(x_1)\mathcal{O}2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C}{|x_1 - x_2|^{2a}|x_1 - x_3|^{2b}|x_2 - x_3|^{2c}} \tag{2.79}$$

where
$$a + b + c = \frac{\Delta_1 + \Delta_2 + \Delta_3}{2}. \tag{2.80}$$

Next, we impose inversion invariance and get

$$2a = \Delta_1 + \Delta_2 - \Delta_3,$$
$$2b = \Delta_1 - \Delta_2 + \Delta_3,$$
$$2c = -\Delta_1 + \Delta_2 + \Delta_3. \tag{2.81}$$

- For four point functions the task becomes more difficult, since it is possible to build conformally invariant cross ratios from four spacetime points:

$$\eta_1 = \frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|}, \qquad \eta_2 = \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|}. \tag{2.82}$$

Generally, four point functions are of the form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = F(\eta_1,\eta_2)\prod_{i<j}^{4}|x_i - x_j|^{\gamma_{ij}} \tag{2.83}$$

where $F(\eta_1,\eta_2)$ can be any function of the cross ratios and

$$\sum_{j\neq i}\gamma_{ij} = -2\Delta_i. \tag{2.84}$$

For higher correlators almost nothing changes. There are more conformally invariant cross ratios, so F becomes a function of more variables, but the principle remains the same. However, we can also show that all higher correlation functions are fixed by two and three point functions. We use the operator product expansion (OPE) in CFTs [32] that allows us to express the product of two operators at x and $y = 0$ as a sum of local operators at y:

$$\mathcal{O}_1(0)\mathcal{O}_2(x) = \sum_i \tilde{C}_i(x)\mathcal{O}_i(0)$$

$$= \sum_{\text{primaries } k} \left(C_k(x)\mathcal{O}_k(0) + C_k^\mu(x)\partial_\mu\mathcal{O}_k(0) + C_k^{\mu\nu}(x)\partial_\mu\partial_\nu\mathcal{O}_k(0) + \ldots \right).$$

(2.85)

In the second line, we have used that the operators in a conformal field theory are the primary operators and their descendants, which can be obtained by acting on the primaries with derivatives.

Since scalar operators transform as $\mathcal{O}_i(x) \to \tilde{\mathcal{O}}(x) = \Lambda^{-\Delta_i}\mathcal{O}_i(\Lambda^{-1}x)$ under a dilatation $x^\mu \to \tilde{x}^\mu(x) = \Lambda x^\mu$, we deduce

$$C_k^{\mu_1\ldots\mu_n}(x) \to \tilde{C}_k^{\mu_1\ldots\mu_n}(x) = \Lambda^{-\Delta_1-\Delta_2+\Delta_k+n}C_k^{\mu_1\ldots\mu_n}(\Lambda^{-1}x). \qquad (2.86)$$

We can fix the $C_k^{\mu_1\ldots\mu_n}$ using conformal symmetry. We express the three point correlator $\langle \mathcal{O}_1(0)\mathcal{O}_2(x)\mathcal{O}_k(y) \rangle$ in two ways. First, we use the expression for the three point function in equation (2.79) and consider the limit $x \to 0$:

$$\langle \mathcal{O}_1(0)\mathcal{O}_2(x)\mathcal{O}_k(y) \rangle = \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}|y|^{\Delta_1+\Delta_k-\Delta_2}|x-y|^{\Delta_2+\Delta_k-\Delta_1}}$$

$$= \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}|y|^{2\Delta_k}}\left(1 - \frac{2x\cdot y}{y^2} + \frac{x^2}{y^2}\right)^{\frac{\Delta_1-\Delta_2-\Delta_k}{2}}$$

$$= \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}|y|^{2\Delta_k}}\left(1 - \frac{\Delta_1-\Delta_2-\Delta_k}{y^2}x\cdot y + O(x^2)\right)$$

(2.87)

Secondly, we use an OPE (2.85) and the two point correlator (2.78) to simplify the three point correlator:

$$\langle \mathcal{O}_1(0)\mathcal{O}_2(x)\mathcal{O}_k(y) \rangle = \sum_{k'} C_{k'}(x)\langle \mathcal{O}_{k'}(0)\mathcal{O}_k(y) \rangle + \sum_{k'} C_{k'}^\mu(x)\langle \partial_\mu\mathcal{O}_{k'}(0)\mathcal{O}_k(y) \rangle \ldots$$

$$= \frac{C_k(x)}{|y|^{2\Delta_k}} + \frac{2C_k^\mu\Delta_k y_\mu}{|y|^{2\Delta_k+2}} + \ldots. \qquad (2.88)$$

Comparing equations (2.87) and (2.88), we find

$$C_k(x) = \frac{C_{12k}}{|x|^{\Delta_1 + \Delta_2 - \Delta_k}}, \tag{2.89}$$

$$C_k^\mu(x) = \frac{C_{12k}}{|x|^{\Delta_1 + \Delta_2 - \Delta_k}} \frac{\Delta_1 - \Delta_2 - \Delta_k}{2\Delta_k} x^\mu, \tag{2.90}$$

$$\vdots$$

$$C_k^{\mu_1 \dots \mu_n}(x) = \frac{C_{12k}}{|x|^{\Delta_1 + \Delta_2 - \Delta_k}} \times (\text{sth. fixed by conformal symmetry}). \tag{2.91}$$

Therefore, we can now recursively use equation (2.85) to reduce any n point function to two point and three point correlators. Consequently, higher correlators are completely fixed by conformal symmetry. Note that we have only considered Poincaré scalars so far. Obviously, finding the correlators for tensorial operators will be harder, but the principle remains the same.

2.2.6 Anomalous scaling dimensions

We have seen in section 2.2.3 that the scaling dimensions of operators in $\mathcal{N} = 4$ SYM in $\mathbb{R}^{1,3}$ are essential quantities since they are the energies in $\mathbb{R} \times S^3$, the boundary of global AdS_5. The time coordinate in the bulk and boundary of AdS_5 agree, so the scaling dimensions on the gauge theory side of the duality also correspond to energies in the gravity theory.

The scaling dimension of an operator gets quantum corrections coming with powers of the effective coupling λ in the gauge theory. Interestingly, these quantum corrections of the scaling do not have to come hand in hand with a non-zero β-function. These corrections even appear in conformal theories such as $\mathcal{N} = 4$ SYM for non-chiral operators. For small λ, we can use perturbation theory to find the quantum corrections to the scaling dimension of operators. For strong coupling, we have to perform the calculation in the corresponding gravity theory.

On the gauge theory side of the duality, we can compute the dimensions of operators in perturbation theory for small λ in the following way: Take all fields $\mathcal{O}_a$ with the same classical dimension Δ_0, which is the mass dimension of the fields. Then, we get

$$\langle \mathcal{O}_a(x)\mathcal{O}_b(y) \rangle = \frac{1}{|x - y|^{2\Delta_0}} \left(\delta_{ab} - \lambda \gamma_{ab} \log(\Lambda^2 |x - y|^2) + \mathcal{O}(\lambda^2) \right) \tag{2.92}$$

for a momentum cutoff Λ. Next, we diagonalize the matrix γ_{ab}. The eigenvectors $\mathcal{O}_a^{\text{diag}}$ are the operators with their corresponding eigenvalues as definite anomalous dimensions $\Delta_a = \Delta_0 + \lambda \gamma_a + \mathcal{O}(\lambda^2)$. We can phrase it differently. Let the dilatation

operator act on the operators $\mathcal{O}_a$ to obtain

$$[D, \mathcal{O}_a(0)] = -i\Delta_{ab}\mathcal{O}_b \tag{2.93}$$

where Δ_{ab} is a square matrix. The number of rows and colums in that matrix is the number of degenerate operators $\mathcal{O}_a$. The $\mathcal{O}_a^{\mathrm{diag}}$ are the eigenvectors of this matrix with corresponding eigenvalues Δ_a. The γ_a are the first order corrections to the anomalous dimensions.

A prominent example of an operator in $\mathcal{N} = 4$ SYM is the Konishi field $\mathcal{O}_K = \mathrm{Tr}\,(\phi_i\phi_i)$. Its classical scaling dimension $\Delta_0 = 2$ is altered by radiative corrections. In the planar limit, these corrections come with powers of the t'Hooft coupling λ and they are known up to high orders [18, 3, 6]. We give the result up to third order:

$$\Delta_{\mathcal{O}_K} = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6} + \mathcal{O}(\lambda^4). \tag{2.94}$$

2.3 Basics of superstring theory

In this section, we briefly introduce some concepts of string theory which are necessary to understand Maldacena's decoupling argument and the derivation of the AdS/CFT correspondence in section 2.5. We explain how supergravity arises as the low energy limit in flat space. Moreover, we introduce the concept of D-branes and their crucial role in modelling gauge theories in string theory. We will see how a $U(N)$ gauge theory arises from a stack of N D-branes. Everything we present in this section can be found in [34].

2.3.1 The Polyakov action

It is well known that we can switch from the most natural choice of an action, the Nambu-Goto action S_{NG}, which measures the area of the worldsheet, to the Polyakov action S_P. This is a transition from a second order to a first order formalism. Let us consider the bosonic sector of the theory:

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \ \sqrt{\det \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}}, \tag{2.95}$$

$$\Rightarrow S_P = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \ \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \tag{2.96}$$

where τ, $0 \leq \sigma < 2\pi$ are the time and space coordinate on the worldsheet, X^μ the spacetime coordinates, $g_{\mu\nu}$ the spacetime metric and $\gamma = \det \gamma_{ab}$. The worldsheet metric γ_{ab} is an auxiliary field whose equations of motion $\delta S/\delta\gamma_{ab} = 0$ must be imposed additionally. If we solve these equations of motion and plug the solution

for the metric in the Polyakov action, we get back the Nambu-Goto action equations of motion.

The Polyakov action has the following symmetries:

- Poincaré invariance in spacetime,
- Weyl invariance on the worldsheet: $\gamma'_{ab}(\tau, \sigma) = e^{2\omega(\tau,\sigma)}\gamma_{ab}(\tau, \sigma)$,
- diffeomorphism invariance on the worldsheet: $\tau' = \tau'(\tau, \sigma), \sigma' = \sigma'(\tau, \sigma) \Rightarrow X'^{\mu}(\tau', \sigma') = X^{\mu}(\tau, \sigma)$.

We see that we have worldsheet symmetries parametrized by three functions: $\tau'(\tau, \sigma), \sigma'(\tau, \sigma)$, and $\omega(\tau, \sigma)$. The metric $\gamma_{ab}(\tau, \sigma)$ also has three degrees of freedom, so we can gauge fix the metric to be

$$\gamma_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.97}$$

In the flat case, $g_{\mu\nu} = \eta_{\mu\nu}$, the equations of motion become

$$\left(\partial_{\sigma}^2 - \partial_{\tau}^2\right) X^{\mu} = 0. \tag{2.98}$$

These are the familiar equations of motion which are solved by left- and rightmoving plane waves on the string. Upon quantisation the different plane wave mode coefficients become the raising and lowering operators for the string states, $\alpha^{\mu}_{-|n|}$ and $\alpha^{\mu}_{|n|}, n = 1, 2, \ldots$, satisfying $[\alpha^{\mu}_m, \alpha^{\nu}_m] = m\delta_{m+n}\eta^{\mu\nu}$. (Technically, we have to normalise and worry about the minus sign in η^{00} to get proper raising and lowering operators.) When we add the fermionic sector of the theory, the tachyonic state is projected out and we get a massless gravity multiplet, $(g_{\mu\nu}, B_{\mu\nu}, \phi)$, as lowest energy (bosonic) excitations of the closed string. All higher bosonic excitations are massive.

2.3.2 String backgrounds

We can write down the action of the string propagating in the background generated by the massless modes $(g_{\mu\nu}, B_{\mu\nu}, \phi)$:

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \ \left[\sqrt{-\gamma}\gamma^{ab}\partial_a X^{\mu}\partial_b X^{\nu} g_{\mu\nu} + \epsilon^{ab}\partial_a X^{\mu}\partial_b X^{\nu} B_{\mu\nu} - \alpha'\sqrt{-\gamma}\mathcal{R}\phi\right]$$
(2.99)

where $\mathcal{R}$ is the Ricci scalar in two dimensions. The last term in this action is just the Euler character

$$\chi = \frac{1}{4\pi} \int d\tau d\sigma \ \sqrt{-\gamma}\mathcal{R} = 2 - 2g, \tag{2.100}$$

which counts the number of holes g (i.e. the genus) in the two dimensional worldsheet. The exponential of the action e^{-S} contains the factor $\left(e^{\phi}\right)^{2(g-1)}$ and since a hole in the string worldsheet is interpreted as a loop in string quantum interactions,
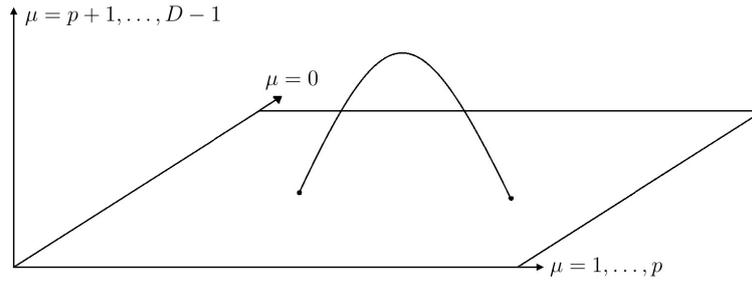
Figure 2.6: A D-brane with a string ending on it. Neumann-Neumann boundary
conditions in the directions $\mu = 0, 1, \ldots, p$, Dirichlet-Dirichlet boundary
conditions in the directions $\mu = p + 1, \ldots, d - 1$.

we can identify $g_s \equiv e^\phi$ as string coupling.

Self-consistency requires the procedure of putting a string in a background to
preserve all symmetries. Imposing Weyl invariance leads to field equations for
$g_{\mu\nu}, B_{\mu\nu}, \phi$. This is how we obtain supergravity as a background. So, the low en-
ergy limit, $\alpha' \to 0$, of 10d superstring theory, which is the theory of massless string
backgrounds, is supergravity in ten dimensions.

2.3.3 D-branes

For open strings the variation of the Polyakov action yields an additional boundary
term,

$$\delta S_{P,\text{boundary}} = -\frac{1}{2\pi\alpha'} \int d\tau \, X^\mu \times \partial^\sigma X_\mu|_{\sigma=0}^{\sigma=2\pi}, \qquad (2.101)$$

which has to vanish to obtain the equations of motion we stated above in equa-
tion (2.98). This provides two possibilities:

- Neumann boundary conditions: $\partial^\sigma X_\mu = 0$ at $\sigma = 0$ and 2π,
- Dirichlet boundary conditions: $\delta X^\mu = 0$ at $\sigma = 0$ and 2π.

We can choose Neumann or Dirichlet boundary conditions separately in every
direction (and also separately for the two ends of the string). For Dirichlet boundary
conditions the end of the string is fixed in the corresponding direction. So, if we
chose Neumann-Neumann boundary conditions (i.e. on both ends of the string) in
p spatial directions and Dirichlet-Dirichlet boundary conditions in the remaining
$d - p - 1$ directions, the endpoints of the string are confined to a hypersurface of
p spatial dimensions in which they can move. In the orthogonal directions, the
endpoints are fixed. Such an object, on which open strings can end, is called a
D-brane where D stands for Dirichlet. We can see an illustration of a D-brane in
figure 2.6.

The spectrum of states of an open string ending on a single D-brane in ten dimen-
sions contains the following massless bosons:

- The parallel excitations $\alpha^i_{-1}|0,p\rangle$, $i = 0,\ldots,p$, where $|0,p\rangle$ is the vacuum with the property $\hat{\Pi}^\mu|0,p\rangle = p^\mu|0,p\rangle$ and $\alpha^\mu_m|0,p\rangle = 0$ for $m > 0$. These form a massless vector, a gauge potential of a $U(1)$ gauge symmetry.
- The orthogonal excitations $\alpha^a_{-1}|0,p\rangle$, $a = p+1,\ldots,9$. These are Goldstone bosons associated with the symmetry breaking of the ten dimensional Poincaré invariance.

We can generalize this to N coincident D-branes. We get the following spectrum on such a stack of branes:

- N^2 massless vectors $\alpha^i_{-1}|jk,p\rangle$, $j,k = 1,2,\ldots,N$ are labels for the branes to which the two ends of the string are attached, called Chan-Paton factors, and
- N^2 sets of massless scalars $\alpha^\mu_{-1}|jk,p\rangle$.

We can write $|jk,p\rangle = \lambda^r_{jk}|r,p\rangle$, $r = 1,2,\ldots,N^2$ such that λ^r form a basis of the $N \times N$ matrices. One can show $\lambda^a = (\lambda^a)^\dagger$, i.e. the matrices span the Lie algebra of $U(N)$. We get a $U(N)$ gauge theory on the worldvolume of the N branes. A comprehensive review of D-branes are the lecture notes [59] by Polchinski.

2.4 Anti-de Sitter space and supergravity

In this section, we give a short introduction to Anti-de Sitter space. For the basic facts see Maldacena's review [48] or the article by Bayona and Braga with many calculational details [5]. The main result of our treatment will be the insight that the boundary of AdS_5 with decompactified time coordinate is $\mathbb{R} \times S^3$. This is, as we have seen in section 2.2.3, conformally equivalent to $\mathbb{R}^4$ where we want to study gauge theories. Furthermore, the isometries of $AdS_5 \times S^5$ are the same as the symmetries of $\mathcal{N} = 4$ SYM which we have found to be $SO(2,4) \times SO(6)$ in section 2.2.2. For more details see, for example, Nastase's review [55].

2.4.1 Embedding and coordinate representations of AdS_{d+1}

AdS_{d+1} can be embedded into $\mathbb{R}^{2,d}$ with coordinates $Y_{-1}, Y_0, Y_1, \ldots, Y_d$ and metric

$$ds^2 = -Y^2_{-1} - Y^2_0 + \sum_{i=1}^d Y^2_i. \tag{2.102}$$

AdS_{d+1} is a hyperboloid in $\mathbb{R}^{2,d}$ defined by

$$-Y^2_{-1} - Y^2_0 + \sum_{i=1}^d Y^2_i = -L^2 \tag{2.103}$$

where L is called the AdS-radius. One way to parametrize this space is given by the global coordinates (ρ, t, Ω_i):

$$Y_{-1} = L \cosh \rho \sin t,$$
$$Y_0 = L \cosh \rho \cos t,$$
$$Y_i = L \Omega_i \sinh \rho \tag{2.104}$$

for $1 \le i \le n$ such that $0 \le \rho$, $0 \le \tau < 2\pi$ and Ω_i are the usual coordinates on the unit d-sphere such that $\sum_i \Omega_i^2 = 1$. By plugging in, we see that in these coordinates the metric becomes

$$ds^2 = L^2 \left(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_{d-1}^2 \right), \tag{2.105}$$

where $d\Omega_{d-1}$ is the usual metric on the unit $(d-1)$-sphere. The factor of $\cosh^2 \rho$ in the (t,t) component of the metric can be interpreted as a gravitational potential confining massive particles to the center of AdS_{d+1} near $\rho = 0$. (However, note that there is no unambiguous way to define the centre, it depends on the choice of coordinates. So, the statement really only says that massive geodesics cannot reach the AdS boundary in a finite amount of time while massless geodesics can.)

The 2π-periodicity of the coordinate t leads to closed timelike curves, as the metric becomes
$$ds^2 \overset{\rho \to 0}{=} L^2(-dt^2 + d\rho^2 + \rho^2 d\Omega_{d-1}^2) \tag{2.106}$$

in the limit $\rho \to 0$. This means AdS_{d+1} becomes $S^1 \times \mathbb{R}^d$ near $\rho = 0$ where S^1 is parametrized by the time coordinate t. We must unwrap this circle by allowing $-\infty < t < \infty$ without identification to prevent causality issues. In topological terms, we move to the universal covering space.

The Poincaré patch coordinates (x^μ, u), $\mu = 0, \dots, d-1$ are defined by

$$u = Y_{-1} + Y_d, \qquad v = Y_{-1} - Y_d, \qquad \frac{u}{L} x^\mu = Y^\mu. \tag{2.107}$$

We can use equation (2.103) to eliminate v. Upon switching to the new coordinates, the metric becomes
$$ds^2 = L^2 \frac{du^2}{u^2} + \frac{u^2}{L^2} dx^\mu dx_\mu \tag{2.108}$$
where $dx^\mu dx_\mu = -(dx^0)^2 + \sum_{i=1}^{d-1}(dx^i)^2$. Replacing $z = \frac{L^2}{u}$ yields

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx^\mu dx_\mu). \tag{2.109}$$

Since the metric becomes singular at $z = 0$ the coordinates only describe either $z > 0$ or $z < 0$. Assuming $z > 0$ implies $Y_{-1} + Y_d > 0$. So, these coordinates do

not cover all AdS_{d+1}, but only the so-called Poincaré patch. We can euclideanize by taking $x_0 \to ix_0$.

2.4.2 Boundary of AdS_{d+1}

In Poincaré coordinates (2.109) the boundary of AdS_{d+1} is Minkowski space $\mathbb{R}^{1,d-1}$ at $z \to 0$ as this corresponds to $r \to \infty$. However, the Poincaré coordinates do not cover all of AdS_{d+1}. We can introduce yet another set of coordinates. We take the metric from equation (2.105) and define

$$r = L \sinh \rho. \tag{2.110}$$

This replacement yields

$$ds^2 = -(L^2 + r^2)dt^2 + \frac{L^2}{L^2 + r^2}dr^2 + r^2 d\Omega_{d-1}^2. \tag{2.111}$$

Factoring out $(L^2 + r^2)$ and further replacing $x = \arctan \frac{r}{L}$ results in

$$ds^2 = -dt^2 + dx^2 + \sin^2 x \, d\Omega_{d-1}^2. \tag{2.112}$$

The boundary at $r \to \infty$, which corresponds to $x = \frac{\pi}{2}$, is $\mathbb{R} \times S^{d-1}$ as the induced metric on the boundary is given by

$$ds_{\partial AdS_{d+1}}^2 = -dt^2 + d\Omega_{d-1}^2. \tag{2.113}$$

So, we have seen that the boundary of Poincaré AdS_{d+1} is $\mathbb{R}^{1,d-1}$ while the boundary of global AdS_{d+1} is $\mathbb{R} \times S^{d-1}$. To show those boundaries are conformally equivalent, we inspect the Euclidean case,

$$d(\log r)^2 + d\Omega_{d-1}^2 = \frac{1}{r^2}(dr^2 + r^2 r^2 d\Omega_{d-1}^2) = \frac{1}{r^2}dx^i dx_i. \tag{2.114}$$

Therefore, the two boundaries are related by a conformal rescaling $r \to \log r$. A Wick rotation shows the equivalence of the Euclidean and Lorentzian case. Consequently, conformal field theories are equivalent in $\mathbb{R}^{1,d}$ and $\mathbb{R} \times S^{d-1}$ since conformal field theories are not affected by conformal mappings. Hence, we can work in Poincaré coordinates even though they do not cover all of AdS_{d+1}.

2.4.3 Symmetries of $AdS_5 \times S^5$

Although there is only an $\mathbb{R} \times SO(1, d - 1)$ symmetry obvious in the Poincaré metric (2.109), the symmetry group of AdS_{d+1} is even bigger. This can be seen in

the definition as an embedding in $\mathbb{R}(2, d)$ in equation (2.103). In this definition, the full symmetry group $SO(2, d)$ is apparent.

$AdS_5 \times S^5$ has an additional $SO(6)$ symmetry from the S^5 factor. The symmetries of $AdS_5 \times S^5$ are therefore $SO(2, 4) \times SO(6)$. We recall that this was the bosonic subgroup of the symmetry group of $\mathcal{N} = 4$ SYM we derived in section 2.2.2. $SO(2, 4) \times SO(6)$ has a six dimensional Cartan subalgebra since both $SO(2, 4)$ and $SO(6)$ are rank 3 groups. We can already infer that each string state in $AdS_5 \times S^5$ is specified by six quantum numbers and there will be an energy E corresponding to the decompactified t direction in AdS_5. Furthermore, there are two spins S_1 and S_2 associated with the $SO(2, 4)$ factor. The $SO(6)$ factor gives rise to three spins J_1, J_2, J_3 corresponding to the string rotating in S^5. So, a string state is

$$|\text{string}\rangle = |E, S_1, S_2; J_1, J_2, J_3\rangle. \tag{2.115}$$

On the gauge theory side also operators and their equivalent states carry six labels. We get the scaling dimension Δ and two spins s_1, s_2 from the $SO(2, 4)$ factor of the bosonic subgroup of the symmetry group. Additionally, there are three R-charges R_1, R_2, R_3 corresponding to the R-symmetry $SO(6)$. Therefore, an operator is given by

$$\mathcal{O} = \mathcal{O}(\Delta, s_1, s_2; R_1, R_2, R_3). \tag{2.116}$$

The matching of the symmetries on both sides of the correspondence is a crucial necessary condition for the ADS/CFT correspondence to hold.

2.4.4 P-branes in supergravity

Loosely speaking, p-branes are black holes extended in p spatial dimensions. They are very massive and curve spacetime around them. They are 1/2 BPS, solitonic solutions of ten dimensional supergravity, i.e. they break one half of the supercharges Q_α. Since we do not want to discuss supergravity in much detail, we refer to Nastase's teaching report [55] or Stelle's review [65].

A p-brane has the symmetries

$$\underbrace{\underbrace{\mathbb{R}^{p+1}}_{\text{transl. on the brane}} \times \underbrace{SO(1, p)}_{\text{Lorentz sym. on the brane}}}_{\text{Poincaré sym. on the brane}} \times \underbrace{SO(9 - p)}_{\text{rot. in transv. dir.}}, \tag{2.117}$$

which are apparent in the following ansatz that solves the supergravity equations of motion:

$$ds^2 = \eta_{\mu\nu} \frac{dx^\mu dx^\nu}{\sqrt{H(r)}} + \sqrt{H(r)} dx^m dx_m \tag{2.118}$$

where $\mu = 0, 1, \ldots p$, $m = p+1, \ldots, 9$. $H(r)$ is a harmonic function of the coordinates x^m, but one can find that it only depends on the distance $r^2 = x^m x_m$:

$$\Delta_{(9-p)} H(x^m) \propto \delta^{(9-p)}(x^m). \tag{2.119}$$

The necessity to recover flat space far away from the brane for $r \to \infty$ requires

$$H(r) = 1 + \left(\frac{L}{r}\right)^{7-p} \tag{2.120}$$

where L is related to the only dimensionful parameter

$$L^{7-p} = N g_s (4\pi)^{(5-p)/2} \Gamma\left(\frac{7-p}{2}\right) \alpha'^{(7-p)/2}. \tag{2.121}$$

For the special case of 3-branes the solution for the supergravity equations of motion is

$$ds^2 = \eta_{\mu\nu} \frac{dx^\mu dx^\nu}{\sqrt{f(r)}} + \sqrt{f(r)} dx^m dx_m \tag{2.122}$$

where $\mu, \nu \in \{0, \ldots, 3\}$, $m \in \{4, \ldots 6\}$, $r^2 = x^m x_m$, $f(r) = 1 + \frac{L^4}{r^4}$ and

$$\frac{L^4}{\alpha'^2} = 4\pi g_s N = \lambda, \tag{2.123}$$

which is the solution for N 3-branes. For $r \gg L$, the metric simplifies to

$$ds^2 \stackrel{r \gg L}{=} \eta_{\mu\nu} dx^\mu dx^\nu + dx_m dx^m \tag{2.124}$$

since

$$\sqrt{1 + \frac{L^4}{r^4}} = 1 + \mathcal{O}\left(\frac{L^4}{r^4}\right), \qquad \frac{1}{\sqrt{1 + \frac{L^4}{r^4}}} = 1 + \mathcal{O}\left(\frac{L^4}{r^4}\right). \tag{2.125}$$

Therefore, we have 10d free supergravity in Minkowski space far away from the brane. For $r \ll L$, the metric becomes

$$ds^2 \stackrel{r \ll L}{=} \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2 \tag{2.126}$$

since

$$\frac{1}{\sqrt{1 + \frac{L^4}{r^4}}} = \frac{r^2}{L^2} + \mathcal{O}\left(\frac{r^4}{L^4}\right), \qquad \sqrt{1 + \frac{L^4}{r^4}} = \frac{L^2}{r^2}\left(1 + \mathcal{O}\left(\frac{r^4}{L^4}\right)\right). \tag{2.127}$$

Let us replace $z = L^2/r$. Then, we find

$$
\begin{aligned}
ds^2 &= \frac{L^2}{z^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{z^2}dz^2 + L^2 d\Omega_5^2 \\
&= \frac{L^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) + L^2 d\Omega_5^2 \\
&= ds^2_{AdS_5} + ds^2_{S^5}.
\end{aligned}
\tag{2.128}
$$

Hence, close to the branes the metric looks like $AdS_5 \times S^5$ with AdS_5 and S^5 having a common radius L.

For the Polyakov action in the $AdS_5 \times S^5$ background, we can factor out the common L^2 and get the prefactor

$$
\frac{L^2}{4\pi\alpha'} = \frac{\sqrt{\lambda}}{4\pi}.
\tag{2.129}
$$

Therefore, $1/\sqrt{\lambda}$ is the parameter counting quantum loops in the string worldsheet theory. Consequently, if the coupling λ in the gauge theory in the planar limit is large, the coupling in string theory is small.

2.5 Maldacena's decoupling argument

One important discovery in the course of the second string revolution in the mid 1990s was Polchinski's insight that D-branes in string theory, i.e. extended objects on which strings can end due to Dirichlet boundary conditions, and p-branes in supergravity are in fact the same thing [58]. This is due to the fact that D-branes carry mass as well as charge and therefore excite bulk gravity modes. This motivates to look at a stack of 3-branes in ten dimensional space from two different points of view. Firstly, we take the open string point of view and consider the theory arising from open strings ending on N D3-branes. Secondly, we will look at the 3-branes as supergravity solutions and study closed strings in the background deformed by the branes. We will see, that in the low energy limit, $\alpha' \to 0$, the theory of open strings on the stack of D-branes splits into two sectors. The same happens to closed strings in the p-brane supergravity background in the low energy limit. Thus, we can motivate the identification of $\mathcal{N} = 4$ SYM and type IIB superstring theory in $AdS_5 \times S^5$.

Let us work out the details. From the open string point of view, D-branes are extended objects in spacetime on which strings can end. As we have seen in section 2.3, N branes give rise to $\mathcal{N} = 4$ SYM $SU(N)$ gauge theory on the worldvolume of the branes in the low energy limit. In this limit, the string mass scale $m_s = \frac{1}{\sqrt{\alpha'}}$ is larger than any occurring excitation. So, the low energy limit corresponds to $\alpha' \to 0$.

Closed strings live in the bulk and bulk excitations interact with the excitations on
the D-branes. For example, two open strings can join to form a closed string and
leave the brane. Hence, the action af the theory has the form

$$S = S_{brane} + S_{int} + S_{bulk}. \tag{2.130}$$

We want the theories on the brane and bulk to decouple in order to split the theory
into two sectors. One can show that the interaction terms are

$$S_{int} \propto g_s \alpha'. \tag{2.131}$$

Therefore, these interactions drop out in the low energy limit $\alpha' \to 0$. Moreover,
in this limit the theories on the brane and in the bulk simplify. On the brane, we
get $\mathcal{N} = 4$ SYM as we have seen in section 2.3. In the bulk, type IIB string theory
reduces to type IIB supergravity.

Based on our treatment of heuristic motivations of the gauge/string duality in
section 2.1, we know that we can approach the same situation from a different
viewpoint. So secondly, consider closed strings in the background given by the
p-brane solution of supergravity seen in section 2.4.4. The background metric is
given by equation (2.122). Therefore, we obtain a redshift factor

$$\sqrt{g_{tt}} = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{4}} \sim \begin{cases} 1, & r \gg L \\ \frac{r}{L}, & r \ll L \end{cases}. \tag{2.132}$$

Consequently, the energy of an object observed at a certain distance $r \ll L$ and at
infinity, E_r and E_∞, are related via

$$\frac{E_r}{E_\infty} = \sqrt{\frac{g_{tt}(\infty)}{g_{tt}(r)}} \stackrel{r \to 0}{=} \frac{L}{r} \quad \Rightarrow \quad E_\infty = \frac{r}{L} E_r. \tag{2.133}$$

Therefore, as an object approaches the D-branes, $r \to 0$, its energy appears to be
lower and lower. Thus, the low energy limit of the theory contains both massless
particles in the bulk and particles close to $r = 0$. In the low energy limit, the two
sectors of the N 3-brane supergravity solution, $r \gg L$ and $r \ll L$, decouple. Bulk
massless excitations are decoupled from the particles close to $r = 0$ since the latter
are trapped by the gravitational potential in the $AdS_5 \times S^5$ region which we have
discovered in the global coordinates (2.105) of AdS-space.

Naively, we only expect the supergravity solution to be valid for the long range
description of branes since we do not expect supergravity to hold on short distances.
For supergravity to be a valid approximation of the string theory in the $AdS_5 \times S^5$

background, its radius must be large in comparison to the string length l_s:

$$L = \sqrt{\alpha'}(4\pi g_s N)^{\frac{1}{4}} \gg \sqrt{\alpha'} = l_s. \qquad (2.134)$$

Since we are in the low energy limit, $\alpha' \to 0$, we must have $\lambda = g_s N \gg 1$. We also need the string coupling g_s to be small, $g_s \to 0$, and therefore $N \to \infty$. For $N \to \infty$, $g_s \to 0$ and $g_s N = \text{const.} \gg 1$, superstring theory in the $AdS_5 \times S^5$ background (2.128) is approximated by classical supergravity.

We can motivate the limit $g_s \to 0$ also from our heuristic description in section 2.1.1 where we have argued that the perturbative $1/N$ expansion on the field theory side corresponds to a genus expansion on the string side with expansion parameter $4\pi g_s = g_{YM}^2 = \frac{\lambda}{N}$. For a sensitive expansion, we need a small expansion parameter g_s.

This leads us to the weakest form of the AdS/CFT conjecture. We identify the two decoupled sectors from both viewpoints. On both sides, one of the sectors is supergravity in ten dimensional flat space. When identifying the two descriptions of the other sector, we claim: $\mathcal{N} = 4$ SYM with gauge group $SU(N)$, $N \to \infty$ in $\mathbb{R}^{1,3}$ in the planar limit with large t'Hooft coupling $\lambda = g_{YM}^2 N \gg 1$ is dual to classical type IIB supergravity in $AdS_5 \times S^5$. In this formulation, we might find disagreements if we go to full string theory, i.e. when λ is not large and, therefore, $l_s \not\ll L$.

AdS/CFT is a strong/weak correspondence. The gauge theory is strongly coupled since the coupling is $\lambda \gg 1$ as we have seen in section 2.1.1. On the string theory side, the worldsheet theory coupling is $\frac{1}{\sqrt{\lambda}}$ as we saw in equation (2.129). In this limit, the string theory is therefore weakly coupled. Perturbation theory on the gauge theory side is only valid for $\lambda \ll 1$, which is the opposite case. But then, the supergravity description on the other side of the correspondence breaks down. This is the reason why the correspondence is called a duality. Both sides of the duality have the same symmetries. The symmetries of AdS_5 agree with the superconformal symmetry of the gauge theory while the symmetries of S^5 agree with the R-symmetry on the gauge theory side.

There are generalisations of the duality which claim that it also holds under less restrictive constraints. A modestly strong form states that the duality holds under $g_s \to 0$, $N \to \infty$, but any fixed λ. This means, since λ is now arbitrary and we are not necessarily in the supergravity limit, we get string theory rather than just supergravity on the side of the closed string point of view. However, since $g_s \to 0$ we still have non-interacting, classical type IIB string theory. On the gauge theory side, we have arbitrary coupling in the planar limit. In particular, the perturbative regime becomes accessible and we can perform calculations using Feynman diagrams. So, the claim of this version of the AdS/CFT correspondence is that $\mathcal{N} = 4$ SYM

	$\mathcal{N} = 4$ SYM	Type IIB strings in $AdS_5 \times S^5$
$N \to \infty$, λ large	planar limit, non-perturbative	classical supergravity
$N \to \infty$, λ small	planar limit, perturbative	classical string theory
N small, λ large	non-planar, non-perturbative	quantum supergravity
N small, λ small	non-planar, perturbative	quantum string theory

Table 2.1: Different limits of the two independent parameters in the AdS/CFT correspondence. Recall that the string parameters are related to N and λ via $4\pi g_s = \lambda/N$ and $L^2/l_s^2 = L^2/\alpha' = \sqrt{\lambda}$.

with gauge group $SU(N)$ in $\mathbb{R}^{1,3}$ in the planar limit is dual to classical type IIB superstring theory in $AdS_5 \times S^5$.

The strongest form of the AdS/CFT correspondence conjectures that $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ in $\mathbb{R}^{1,3}$ is exactly equivalent to full quantum type IIB superstring theory in $AdS_5 \times S^5$. Here, both g_s and N are arbitrary. However, it is not possible to test this to date since we do not know how to perform non-perturbative quantization of string theory in general. An overview of the different limits and their effects on the gauge theory and the string theory side can be found in table 2.1.

Since the boundary of AdS_5 in global coordinates is $\mathbb{R} \times S^3$, we can think of the gauge theory being located on the boundary of the Anti-de Sitter space. The time coordinate in the bulk and on the boundary agree as we have seen in section 2.2.6. Furthermore, we have seen in equation (2.53) that the energies in a conformal field theory in $\mathbb{R} \times S^3$ correspond to scaling dimensions in $\mathbb{R}^{1,3}$. This is why we can conclude that for operators $\mathcal{O}_A$ in $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ and corresponding string states $|\mathcal{O}_A\rangle$ in $AdS_5 \times S^5$ we have

$$\langle \mathcal{O}_A(x)\mathcal{O}_B(y)\rangle = \frac{C\delta_{A,B}}{(x-y)^{2\Delta_A(\lambda,\frac{1}{N})}} \quad \Leftrightarrow \quad \mathcal{H}_{\text{string}}|\mathcal{O}_A\rangle = E_A\left(\frac{R^2}{\alpha'}, g_s\right)|\mathcal{O}_A\rangle \quad (2.135)$$

where

$$\Delta_A\left(\lambda, \frac{1}{N}\right) = E_A\left(\frac{L^2}{\alpha'}, g_s\right) \quad (2.136)$$

if the parameters are identified in the way we have seen before:

$$\lambda = \frac{L^4}{\alpha'^2}, \qquad \frac{\lambda}{N} = 4\pi g_s. \quad (2.137)$$

In the following we will use equation (2.136) to check the correspondence by calculating the energies and scaling dimensions on the two sides and compare them.

3 The BMN limit and plane-wave string/gauge theory duality

In 2002, Berenstein, Maldacena and Nastase (BMN) derived in an ifluential paper that $\mathcal{N} = 4$ SYM and type IIB string theory have an overlapping perturbative regime, in which results of calculations on both sides of the duality can be compared [16]. They established the so-called plane-wave string/gauge theory duality by showing that operators with high R-charges on the gauge theory side of the AdS/CFT duality correspond to strings in a plane-wave background. In the limit of large R-charges a new perturbative expansion parameter λ' arises on both sides of the duality. Significantly, the plane-wave string/gauge theory duality is not a strong/weak correspondence. Since the same parameter appears on both sides, the weak and strong coupling regimes of the two theories coincide. The key idea is that for a string solution with large R-charge J we get

$$E - J = E_2(\lambda') + \frac{1}{\sqrt{J}} E_4(\lambda') + \ldots \stackrel{J \to \infty}{\Rightarrow} E_2(\lambda') \tag{3.1}$$

So in the large J limit with λ' constant, we see that the lowest order approximation becomes exact. The reason for this simplification on the string theory side of the correspondence is that a particle moving very fast on the S^5 component of $AdS_5 \times S^5$ effectively sees a plane-wave background, the so-called Penrose limit of $AdS_5 \times S^5$. Therefore, strings with high R-charges will live in a simplified background which leads to a quadratic action. Metsaev and Tseytlin [51, 52] found that we can quantize strings in this background, unlike general strings in $AdS_5 \times S^5$, by going to lightcone coordinates. Consequently, we can explore the string side of the duality beyond the supergravity limit as we know how to handle real stringy excitations. On the gauge theory side the Penrose limit corresponds to a new non-t'Hooftian double scaling limit, called BMN limit, which sends $\lambda \to \infty$, $J \to \infty$ while keeping λ/J^2 constant. An accessible and comprehensive review of the subject was written by Plefka [57].

In the first section of this chapter, , we describe how the plane-wave geometry arises from $AdS_5 \times S^5$ in the Penrose limit. This explains why particles with a large R-charge effectively live in a plane-wave background.

In section 3.2, we introduce two conserved charges p^+ and p^-, the lightcone mo-

menta, which are a natural choice in lightcone coordinates. We derive their relation to the equivalent set of charges given by the energy E and the R-charge J. Based on those new charges we can judge which states will be present after taking the Penrose limit and which part of the spectrum will be projected out.

In section 3.3, we translate the effect of the Penrose limit to the gauge theory operators. We can conclude what kind of limit we have to take on the gauge theory side to project out the same parts of the spectrum. The resulting limit on the gauge theory side is, as we already metioned, called the BMN limit.

In section 3.4, we quantize type IIB string theory in a plane-wave background along the lines of Metsaev and Tseytlin [51, 52]. We will focus on the bosonic sector and stress the conceptual similarities to the lightcone quantization of superstrings in flat space [34]. We will arrive at having complete knowledge about the perturbative quantum theory of strings in the plane-wave background, i.e. we know all states and their energies.

In section 3.5, we will match the string theory states we have found to operators in the $\mathcal{N} = 4$ SYM. We check agreement between the string theory energies and the gauge theory scaling dimensions. We will see that the scaling dimensions come as a perturbative expansion in the new effective coupling $\lambda' = \lambda/J^2$. In order to check the agreement beyond the small λ' case, we must perform a resummation of graphs from all orders of the t'Hooft expansion to obtain an all-loop result for the scaling dimension on the gauge theory side.

In the last section of this chapter, section 3.6, we consider a specific example to show how graphs of all genera contribute to the resummation from the previous section. We also present how the effective parameter $g_2 = J^2/N$ arises as an effective genus expansion parameter.

3.1 The Penrose limit of $AdS_5 \times S^5$

Using the metric of AdS_5 in global coordinates (2.105), we can write the metric of $AdS_5 \times S^5$ as

$$ds^2_{Ads5 \times S^5} = L^2\big(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2$$
$$+ \cos^2\theta\, d\psi^2 + d\theta^2 + \sin^2\theta\, d\Omega_3'^2\big). \tag{3.2}$$

We want to find the easiest string solution with high R-charge. Firstly, this means that we restrict ourselves to a pointlike string solution, i.e. the spacetime coordinates of the string only depend on the worldsheet time τ, but not spacelike worldsheet coordinate σ. Secondly, the string is stationary in the centre of the AdS_5 component of spacetime. Thirdly, the pointlike string moves on a great circle of the S^5 component

and by Symmetry we can locate the great circle in the equatorial plane. Altogether, we consider a massless particle stationary in the centre of AdS_5 and moving on a great circle in the S^5. The lightlike trajectory of the particle, parametrized by τ, is given by

$$t = t(\tau), \quad \psi = \psi(\tau), \quad \rho = 0 \quad \text{and} \quad \theta = 0. \tag{3.3}$$

The equations of motion, derived using the Polyakov action in $AdS_5 \times S^5$, are

$$\dot{t}^2 = \dot{\psi}^2. \tag{3.4}$$

Here, the dot denotes a derivative with respect to τ. We define $\tilde{x}^{\pm} = \frac{1}{2}(t \pm \psi)$ and then introduce the coordinates

$$x^+ = \frac{\tilde{x}^+}{\mu}, \quad x^- = \mu L^2 \tilde{x}^-, \quad \rho = \frac{r}{L} \quad \text{and} \quad \theta = \frac{y}{L}. \tag{3.5}$$

The parameter μ will be useful later to examine the flat space limit. Clearly, the lightlike trajectory

$$\tilde{x}^- = \tau, \quad \tilde{x}^+ = c = \text{const.} \quad \Leftrightarrow \quad t = c + \tau, \quad \psi = c - \tau \tag{3.6}$$

solves the equations of motion (3.4).

We transform the metric (3.2) to the coordinates defined above in equation (3.5). We are interested in the metric close to the lightlike trajectory, so we choose the limit $L \to \infty$, the so-called Penrose limit. Using

$$\cosh^2 \rho = 1 + \frac{r^2}{L^2} + \mathcal{O}\left(L^{-4}\right), \quad \cos^2 \theta = 1 - \frac{y}{L^2} + \mathcal{O}\left(L^{-4}\right) \tag{3.7}$$

and

$$t = \mu x^+ + \frac{1}{\mu L^2} x^-, \quad \psi = \mu x^+ - \frac{1}{\mu L^2} x^- \tag{3.8}$$

to transform the metric to the new coordinates and to take the scaling limit $L \to \infty$, we obtain

$$
\begin{aligned}
ds^2_{AdS_5 \times S^5} ={}& L^2 \left[-\left(1 + \frac{r^2}{L^2} + \mathcal{O}\left(L^{-4}\right)\right) \left(\mu dx^+ + \frac{1}{\mu L^2} dx^-\right)^2 \right. \\
&+ \frac{1}{L^2} dr^2 + \left(\frac{r^2}{L^2} + \mathcal{O}\left(L^{-4}\right)\right) d\Omega_3^2 \\
&+ \left(1 - \frac{y^2}{L^2} + \mathcal{O}\left(L^{-4}\right)\right) \left(\mu dx^+ - \frac{1}{\mu L^2} dx^-\right)^2 \\
&\left. + \frac{1}{L^2} dy^2 + \left(\frac{y^2}{L^2} + \mathcal{O}\left(L^{-4}\right)\right) d\Omega_3'^2 \right] \\
={}& L^2 \left[-\mu^2 (dx^+)^2 + \mu^2 (dx^+)^2 \right] + \left[-2 dx^+ dx^- + \mu^2 r^2 (dx^+)^2 + dr^2 + r^2 d\Omega_3^2 \right.
\end{aligned}
$$

$$- 2dx^+dx^- - \mu^2 y^2 (dx^+)^2 + dy^2 + y^2 d\Omega_3'^2] + \mathcal{O}(L^{-2})$$
$$= - 4dx^+dx^- - \mu^2(\boldsymbol{y}^2 + \boldsymbol{r}^2)(dx^+)^2 + d\boldsymbol{y}^2 + d\boldsymbol{r}^2 + \mathcal{O}(L^{-2}). \qquad (3.9)$$

We have introduced four new coordinates $\boldsymbol{y} = (y^1, \ldots, y^4)$ and $\boldsymbol{r} = (r^1, \ldots, r^4)$ using

$$d\boldsymbol{r}^2 = dr^2 + r^2 d\Omega_3{}^2, \qquad\qquad \boldsymbol{r}^2 = r^2, \qquad\qquad (3.10)$$

which is the Euclidean metric in Cartesian coordinates on the left hand side and spherical coordinates on the right hand side. The equivalent statement is true for $\boldsymbol{y}$. Taking the limit $L \to \infty$ in equation (3.9), we see that the $AdS_5 \times S^5$ metric (3.2) becomes the plane-wave metric,

$$ds^2_{AdS_5 \times S^5} \to ds^2_{\text{pw}} = -4dx^+dx^- - \mu^2(x^i)^2(dx^+)^2 + (dx^i)^2, \qquad (3.11)$$

in the Penrose limit. We introduced the coordinates x^i, $i = 1, \ldots, 8$, to replace $r^i = x^i$ and $y^i = x^{i+4}$ for $i = 1, \ldots, 4$. For $\mu \to 0$, equation (3.11) becomes the Minkowski metric. Consequently, the auxiliary parameter μ controls the flat space limit.

3.2 Lightcone momenta

In the coordinates (3.2), the energy is the conserved charge corresponding to time translations, i.e. it is given by $E = i\partial_t$. The angular momentum is $J = -i\partial_\psi$. The two charges are conserved because the metric does not explicitly depend on t and ψ. Hence, t and ψ are cyclic coordinates of the Lagrangian associated with the Nambu-Goto or Polyakov action. To translate these two charges to the dual gauge theory we recall that the gauge theory lives on the $\mathbb{R} \times S^3$ boundary of AdS_5. Because the time coordinates in the bulk and boundary are the same, the energy in the bulk corresponds to the energy on the boundary. However, we have seen in section 2.2.3 that the conformal mapping from $\mathbb{R} \times S^3$ to $\mathbb{R}^{1,3}$ identifies time translations on the cylinder to scale transformations on the plane. So the energy E of a string state corresponds to the scaling dimension Δ of the corresponding operator in $\mathcal{N} = 4$ SYM. The rotations in the S^5 factor correspond to R-symmetry transformations in the gauge theory. This can be inferred from comparing the symmetry groups. The symmetry transformations of S^5 are associated to the $SO(6)$ factor of the $AdS_5 \times S^5$ symmetry. The R-symmetry corresponds to the same factor in the bosonic subgroup of the symmetry group of $\mathcal{N} = 4$ SYM. Hence, J corresponds to an R-charge on the gauge theory side of the duality.

Since both t and ψ are cyclic coordinates, x^+ and x^- are cyclic, too. The two

associated conserved charges are equivalent to E and J. We can relate E and J to the conjugate momenta of $x^\pm$. The plane-wave metric is

$$g_{\mathrm{pw}} = \begin{pmatrix} -\mu^2(x^i)^2 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{8\times8} \end{pmatrix} \quad \Rightarrow \quad g_{\mathrm{pw}}^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{\mu^2(x^i)^2}{4} & 0 \\ 0 & 0 & \mathbf{1}_{8\times8} \end{pmatrix}. \qquad (3.12)$$

Therefore, the conjugate momenta are

$$\begin{aligned} \mathcal{H}_{\mathrm{lc}} \equiv 2p^- &= 2g^{--}p_- + 2g^{-+}p_+ \\ &= -\frac{\mu^2(x^i)^2}{2}(-i\partial_-) - (-i\partial_+) \\ &= \frac{\mu^2(x^i)^2}{2}\frac{i}{\mu L^2}(\partial_t - \partial_\psi) + i\mu(\partial_t + \partial_\psi) \\ &= i\mu(\partial_t + \partial_\psi) + \mathcal{O}(L^{-2}) \\ &= \mu(E - J) + \mathcal{O}(L^{-2}), \qquad (3.13) \\ 2p^+ = 2g^{+-}p_- &= \frac{i}{\mu L^2}(\partial_t - \partial_\psi) \\ &= \frac{E + J}{\mu L^2}. \qquad (3.14) \end{aligned}$$

In equation (3.13), $\mathcal{H}_{\mathrm{lc}}$ is the lightcone Hamiltonian. Comparing the above expression for this Hamiltonian to equation (3.1) from the introduction, we see that the large R-charge limit simplifies $E - J$ of a string state to E_{lc}/μ. This quantity E_{lc}/μ is the exact difference of the energy and the angular momentum of a state in the large J limit. In section 3.5, we will see that it only depends on a new parameter λ', which is finite in the Penrose limit. It is clear from equation (3.14) that generic string states have vanishing p^+ in the Penrose limit $L \to \infty$ unless J and L are correlated as

$$J \sim L^2. \qquad (3.15)$$

For the lightcone Hamiltonian $\mathcal{H}_{\mathrm{lc}}$ to remain finite in the Penrose limit, we can conclude from

$$\mathcal{H}_{\mathrm{lc}} = \mu(E - J) \quad \text{that} \quad E \approx J \qquad (3.16)$$

since otherwise $\mathcal{H}_{\mathrm{lc}}$ diverges.

3.3 The Penrose limit in the dual gauge theory

We want to know what the Penrose limit $L \to \infty$ corresponds to in the $\mathcal{N} = 4$ SYM in order to translate it from the string theory to the gauge theory. In section 3.1 we have seen that close to the geodesic of a pointlike string moving on great circles in

S^5 the metric simplifies to the plane-wave metric. Therefore we expect a simplified string spectrum, which we will confirm in section 3.4. In order to compare the string theory results to gauge theory, we have to take the same limit on the gauge theory side, i.e. the same part of the spectrum needs to be projected out on both sides of the duality. We use the two crucial relations (2.24) and (2.123) that connect the parameters on the gauge and string theory side,

$$\frac{L^4}{\alpha'^2} = g_{YM}^2 N, \qquad 4\pi g_s = g_{YM}^2. \tag{3.17}$$

In equation (3.15), we saw that the R-charge has to obey $J \sim L^2$. Furthermore, the string coupling g_s is not affected by the Penrose limit and is therefore to be held fixed. The two relations in equation (3.17) then tell us that the Penrose limit translates to the gauge theory according to

$$N \to \infty, \qquad J \sim \sqrt{N} \qquad \text{and} \qquad g_{YM} = \text{fixed}. \tag{3.18}$$

Let us stress that this limit is different from the standard planar t'Hooft limit (2.22). In the planar limit, we have $g_{YM} \to 0$ while $N \to \infty$ such that $\lambda = g_{YM}^2 N$ is fixed. In contrary, the t'Hooft coupling λ diverges in the limit (3.18) we consider here.

Translating the right hand side of equation (3.13) to the gauge theory side leads to the crucial correspondence in the plane-wave string/gauge theory duality:

$$\frac{\mathcal{H}_{\text{lc}}}{\mu} \,\hat{=}\, \Delta - J. \tag{3.19}$$

From equation (3.16) we get that only operators with

$$\Delta \approx J \tag{3.20}$$

will remain after taking the Penrose limit since only those correspond to finite light-cone energy states.

3.4 Quantization of the type IIB plane-wave superstring

When quantizing the string in the plane-wave background (3.11), we have to consider that the plane-wave supergravity solution is supported by a Ramond-Ramond (RR) 5-form flux [19]. This RR background field forces us to work in the Green-Schwarz formulation defined by the worldsheet fields $x^\mu(\tau, \sigma)$ and $\theta_\alpha^A(\tau, \sigma)$, a 10d vector and two Majorana-Weyl spinors with $\mu = 0, \ldots, 9$, $A = 1, 2$ and $\alpha = 1, \ldots, 16$.

The quantization of strings in this background was accomplished by Metsaev and Tseytlin [51, 52] in 2002 and is most conveniently performed in lightcone gauge. In this gauge choice, the problem is dramatically simplified. A precise treatment of the procedure is too extensive for this dissertation, so we refer to the given references for details. We restrict ourselves mainly to the bosonic sector and focus on the crucial steps. First, we note that the process is along the lines of quantizing the superstring in a flat background in lightcone gauge [34]. In the limit $\mu = 0$, we can always restrict to the flat case and we should therefore be able to reproduce the known results.

We first use diffeomorphism invariance of the Polyakov action to go to conformal gauge. In consequence, the worldsheet metric γ_{ab} is given by

$$\gamma_{ab} = e^{\phi}\eta_{ab}. \tag{3.21}$$

Residual conformal invariance allows us to set $x^+(\tau, \sigma) = p^+\tau$. After using κ-symmetry to gauge away half of the fermionic degrees of freedom as in the flat case, we obtain a free, quadratic action

$$S = \frac{1}{2\pi\alpha'}\int d\tau d\sigma \left[\frac{1}{2}(\partial_a x^i)^2 - \frac{1}{2}m^2(x^i)^2 + i\theta^1(\partial_\tau + \partial_\sigma)\theta^1 \right.$$
$$\left. + i\theta^2(\partial_\tau - \partial_\sigma)\theta^2 - 2m\theta^1\Gamma_{1234}\theta^2\right] \tag{3.22}$$

with $i = 1, \ldots, 8$ as before. We set $m = \mu p^+$ to simplify the notation. The bosonic mass term, for example, comes from

$$g_{++}\eta^{ab}\partial_a x^+ \partial_b x^+ = -\mu^2(x^i)^2(\partial_\tau(p^+\tau))^2 = -\mu^2(p^+)^2(x^i)^2 = -m^2(x^i)^2. \tag{3.23}$$

The fermionic mass term arises from the coupling of the fermions to the RR 5-form. We see that we get the action for eight massive bosons and two massive fermions. For the bosons, the equations of motion become

$$(\partial_\tau^2 - \partial_\sigma^2 + m^2)x^i = 0 \tag{3.24}$$

subject to the closed string boundary condition $x^i(\tau, \sigma + 1) = x^i(\tau, \sigma)$ where we have chosen the length of the string to be 1. We clearly see that the flat case arises if we set $m = 0$.

In order to absorb the dependence on the string tension for convenience (see [52]), we perform a rescaling that involves $x^+ \to x^+$, $x^- \to 2\pi\alpha'x^-$, $x^i \to \sqrt{2\pi\alpha'}x^i$ and $\theta^i \to \sqrt{2\pi\alpha'}\theta^i$. The dependence is restored by taking $p^+ \to 2\pi\alpha'p^+ \Rightarrow m \to 2\pi\alpha'm$.

Furthermore, the metric (3.11) becomes

$$ds_{\mathrm{pw}} \to 2\pi\alpha' ds_{\mathrm{pw}}. \tag{3.25}$$

Therefore, the action (3.22) does not depend on the string tension any more. The equations of motion (3.24) are then solved by an oscillator mode decomposition ansatz which is very similar to the flat case,

$$x^i(\sigma,\tau) = x_0^i \cos m\tau + \frac{p_0^i}{m}\sin m\tau + \sum_{n\neq 0}\frac{i}{\omega_n}\left(\alpha_n^i e^{-i(\omega_n\tau - k_n\sigma)} + \tilde{\alpha}_n^i e^{-i(\omega_n\tau + k_n\sigma)}\right), \tag{3.26}$$

where $\omega_n = \mathrm{sign}(n)\sqrt{k_n^2 + m^2}$ and $k_n = 2\pi n$. A similar ansatz is made for the fermions. Since the complex conjugate of equation (3.26) is

$$x^{\dagger i} = x_0^{\dagger i}\cos m\tau + \frac{p_0^{\dagger i}}{m}\sin m\tau + \sum_{n\neq 0}\frac{-i}{\omega_n}\left(\alpha_n^{\dagger i}e^{i(\omega_n\tau - k_n\sigma)} + \tilde{\alpha}_n^{\dagger i}e^{i(\omega_n\tau + k_n\sigma)}\right)$$

$$= x_0^{\dagger i}\cos m\tau + \frac{p_0^{\dagger i}}{m}\sin m\tau + \sum_{n\neq 0}\frac{i}{\omega_{-n}}\left(\alpha_n^{\dagger i}e^{-i(\omega_{-n}\tau - k_{-n}\sigma)} + \tilde{\alpha}_n^{\dagger i}e^{-i(\omega_{-n}\tau + k_{-n}\sigma)}\right), \tag{3.27}$$

reality of the x^i, $x^{\dagger i} = x^i$, implies that

$$x_0^{\dagger i} = x_0^i, \qquad p_0^{\dagger i} = p_0^i,$$
$$\alpha_n^{\dagger i} = \alpha_{-n}^i, \qquad \tilde{\alpha}_n^{\dagger i} = \tilde{\alpha}_{-n}^i. \tag{3.28}$$

Now, we follow the usual procedure of canonical quantization known from the flat case [34]. Based on the mode expansion ansätze for the bosons and fermions, we can calculate the corresponding conjugate momenta, e.g.

$$p^i = \dot{x}^i = p_0^i \cos m\tau - mx_0^i \sin m\tau + \sum_{n\neq 0}\left(\alpha_n^i e^{-i(\omega_n\tau - k_n\sigma)} + \tilde{\alpha}_n^i e^{-i(\omega_n\tau + k_n\sigma)}\right), \tag{3.29}$$

and impose canonical commutation and anticommutation relations to get the algebra of the mode expansion coefficients $\alpha_n^i, \tilde{\alpha}_n^i$ and the fermionic equivalents. For the bosonic sector, this procedure yields

$$[p_0^i, x_0^j] = -i\delta^{ij}, \qquad\qquad [\alpha_m^i, \tilde{\alpha}_n^j] = 0,$$

$$[\alpha_m^i, \alpha_n^j] = [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = \frac{1}{2}\omega_m \delta_{m+n,0}\delta^{ij} \tag{3.30}$$

with $i, j = 1, \ldots 8$ and $m, n \neq 0$. We introduce the operators

$$a_n^i = \sqrt{\frac{2}{|\omega_n|}} \alpha_n^i \qquad \text{and} \qquad \tilde{a}_n^i = \sqrt{\frac{2}{|\omega_n|}} \tilde{\alpha}_n^i. \tag{3.31}$$

For $n \geq 1$ the $a_{-n}^i = a_n^{\dagger i}$ are creation operators and the a_n^i are annihilation operators since

$$[a_n^i, a_{-n}^j] = \delta^{ij} \qquad \text{for } n \geq 1. \tag{3.32}$$

Moreover, for the zero modes we rewrite

$$a_0^i = \frac{1}{2m}(p_0^i - imx_0^i). \tag{3.33}$$

Consequently, we get the commutator

$$[a_0^i, a_0^{\dagger j}] = \delta^{ij}, \tag{3.34}$$

which shows that the $a_0^{\dagger i}$ are creation operators and the a_0^i are annihilation operators.

We can write the lightcone Hamiltonian from equation (3.13) as

$$\mathcal{H}_{\text{lc}} = \frac{1}{p^+} \int_0^1 d\sigma \, \left[(p^i)^2 - (\partial_\sigma x^i)^2 + m^2(x^i)^2 + \text{fermions}\right]$$

$$= \mu(a_0^i)^\dagger a_0^i + \frac{1}{\alpha' p^+} \sum_{n=1}^{\infty} \sqrt{n^2 + (\alpha' p^+ \mu)^2} \, \big(\underbrace{a_{-n}^i}_{a_n^{\dagger i}} a_n^i + \underbrace{\tilde{a}_{-n}^i}_{\tilde{a}_n^{\dagger i}} \tilde{a}_n^i \big) + \text{fermions}. \tag{3.35}$$

Here, we can identify the number of bosons in the nth mode,

$$N_n^b \equiv \alpha_n^{\dagger i} \alpha_n^i. \tag{3.36}$$

If we let N_n^f be the equivalent number operator for fermions, the full expression for the lightcone Hamiltonian turns out to be

$$\mathcal{H}_{\text{lc}} = \mu(N_0^b + N_0^f) + \frac{1}{\alpha' p^+} \sum_{n=1}^{\infty} \sqrt{n^2 + (\alpha' p^+ \mu)^2} \, \left(N_n^b + \tilde{N}_n^b + N_n^f + \tilde{N}_n^f \right)$$

$$= \mu N_0 + \mu \sum_{n=1}^{\infty} \sqrt{1 + \frac{n^2}{(\alpha' p^+ \mu)^2}} (N_n + \tilde{N}_n) \tag{3.37}$$

where $N_n^f, \tilde{N}_n^f$ count the two types of fermions and $N_n = N_n^b + N_n^f$, $\tilde{N}_n = \tilde{N}_n^b + \tilde{N}_n^f$.

Finally, the Virasoro constraints coming from the equations of motion of the aux-

iliary worldsheet metric amount to the physical state condition

$$(N - \tilde{N})|\text{phys}\rangle = 0, \qquad \text{where} \quad N = \sum_{n=1}^{\infty}(N_n^b + N_n^f)$$

$$\text{and} \quad \tilde{N} = \sum_{n=1}^{\infty}(\tilde{N}_n^b + \tilde{N}_n^f). \tag{3.38}$$

In conclusion, we can state that we have full control over the spectrum of the free superstring in the plane-wave background supported by a RR 5-form. The vacuum state $|0\rangle$ is defined as being annihilated by all bosonic annihilation operators a_0^i, a_n^i, $\tilde{a}_n^i$, with $n \geq 1$ and by all the fermionic ones. All states are formed by acting with creation operators $a_0^{\dagger i}$, a_{-n}^i, $\tilde{a}_{-n}^i$, with $n \geq 1$ and the ferionic equivalents on the vacuum respecting the physical state condtition (3.38). The energies of these states are then given by

$$E_{\text{lc}} = \mu N_0 + \mu \sum_{n=1}^{\infty} \sqrt{1 + \frac{n^2}{(\alpha' p^+ \mu)^2}}(N_n + \tilde{N}_n). \tag{3.39}$$

The next steps are obvious now that we have full knowledge about the string spectrum in the Penrose limit. We will try to identify the gauge theory operators corresponding to the string theory states. If we manage to compute the scaling dimensions Δ of the operators, we can compare them to the lightcone energies E_{cl} of the dual string states via the relation (3.19), which states

$$\frac{E_{\text{lc}}}{\mu} = \Delta - J. \tag{3.40}$$

3.5 Plane-wave strings from $\mathcal{N} = 4$ SYM

In section 3.3, we have seen that the Penrose limit corresponds to (3.18),

$$N \to \infty \quad \text{and} \quad J \to \infty \quad \text{with} \quad \frac{J^2}{N} \text{ and } g_{YM} \text{ fixed} \tag{3.41}$$

in the dual gauge theory. On this side of the duality, we call it BMN limit. We have also seen that, in order to have finite lightcone momentum $p^- = \frac{1}{2}\mathcal{H}_{\text{lc}}$ in the BMN limit, we must require equation (3.20),

$$\Delta \approx J, \tag{3.42}$$

to hold for the dual gauge theory operators. We recall from section 2.2.1 that the dual gauge theory, $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ in 4d, contains a

vector A^μ, four Majorana fermions ψ^a, $\overline{\psi}_a$ and six scalars ϕ_i, $i = 1, \ldots, 6$. The conserved charge J is an R-charge, one of the three Cartan generators of the $SO(6)$ R-symmetry, which we have examined in section 2.2.2. The R-symmetry rotates the six scalars ϕ_i among themselves and also the four spinors as $SO(6) \cong SU(4)$. We chose J to rotate around the equator of S^5 which corresponds to the $SO(2) \subset SO(6)$ subgroup rotating only ϕ_5 and ϕ_6 in the gauge theory.

Since we are interested in operators with $\Delta - J \approx 0$, we first consider the operator with $\Delta - J = 0$. This operator is $\mathrm{Tr}\,[Z^J]$ where $Z = \phi^5 + i\phi^6$. Since this is a chiral primary, its scaling dimension is protected from quantum corrections by supersymmetry [26, 1, 2]. This stability under quantum corrections makes chiral primary operators crucial in the BMN plane-wave string/gauge theory duality. We have seen in section 2.2.6 that the scaling dimension gets quantum corrections coming with powers of λ. Therefore, it seems useless to consider the BMN limit as it includes $\lambda \to \infty$, which makes quantum corrections absolutely uncontrollable. Nevertheless, since chiral primaries remain uncorrected, their scaling dimension in the BMN limit is equal to the classical dimension and they can be conformal primaries or descendants. For example, in the scalar sector the chiral primaries are

$$\mathcal{O}_{CP}^k = C_{i_1 i_2 \ldots i_k} \mathrm{Tr}\,[\phi_{i_1} \phi_{i_2} \ldots \phi_{i_k}] \tag{3.43}$$

where $C_{i_1 i_2 \ldots i_k}$ is symmetric and traceless. The scaling dimension of an operator $\mathcal{O}_{CP}^k$ is

$$\Delta_{\mathcal{O}_{CP}^k} = k, \tag{3.44}$$

which is simply its classical mass dimension. Even though the chiral primary operators have such a remarkably convenient property, Berenstein, Maldacena and Nastase did not restrict themselves to these. Their crucial insight was, that we can perturb chiral primaries in a controlled way such that the resulting operators still receive small quantum corrections in the BMN limit.

Let us first explore the available operators that satisfy (3.42), $\Delta \approx J$. We build operators out of the building blocks in table 3.1. $\mathrm{Tr}\,[Z^J]$ is of the form (3.43), satisfies (3.42) and, therefore, is a chiral primary surviving the BMN limit. By inserting a small number of operators with $\Delta - J = 1$ in a string of Z's we form so-called BMN operators, e.g.

$$\mathrm{Tr}\,[D_\mu Z \ldots Z \phi_i Z \ldots Z D_\nu Z \ldots Z \psi_A Z \ldots Z], \tag{3.45}$$

These are not always chiral primaries. We will see at the end of this section that BMN operators receive quantum corrections coming with powers of the counting parameter

$$\lambda' = \frac{g_{YM}^2 N}{J^2}, \tag{3.46}$$

	ϕ_i	A_μ	Z	$\overline{Z}$	ψ_A	$\overline{\psi}_{\dot{A}}$
Δ_0	1	1	1	1	3/2	3/2
J	0	0	1	-1	1/2	-1/2
$\Delta_0 - J$	1	1	0	2	1	2

Table 3.1: Classical scaling dimensions and R-charges of fundamental building blocks of operators in $\mathcal{N}=4$ SYM. Here, $i=1,\ldots,4$, $\mu=1,\ldots,4$ and $A=1,\ldots,8$.

which is finite in the BMN limit. Therefore, we have perturbed the chiral primary $\mathrm{Tr}\,[Z^J]$ in a controlled way, such that its scaling dimension is still well bahaved in the BMN limit.

Let us use the crucial relation from equation (3.19),

$$\frac{\mathcal{H}_{\mathrm{lc}}}{\mu} \,\hat{=}\, \Delta - J, \tag{3.47}$$

in order to match states in the plane-wave string theory with operators in the gauge theory. We have already identified the unique operator with $\Delta - J = 0$, $\mathrm{Tr}\,[Z^J]$, which corresponds to the plane-wave string vacuum state $|0, p^+\rangle$ satisfying

$$\mathcal{H}_{\mathrm{lc}}|0, p^+\rangle = 0. \tag{3.48}$$

The operators satisfying $\Delta - J = 1$ correspond to string states with $E_{\mathrm{lc}} = \mu$ according to equation (3.47). We derived three types of string excitations with this energy in section 3.4. We match these excitations with operators in the gauge theory:

$$a_0^{\dagger i}|0, p^+\rangle \quad \hat{=} \quad \frac{1}{\sqrt{N^J}}\mathrm{Tr}\,(\phi_i Z^J) \qquad \text{for } i = 5, \ldots, 8, \tag{3.49}$$

$$a_0^{\dagger \mu}|0, p^+\rangle \quad \hat{=} \quad \frac{1}{\sqrt{N^J}}\mathrm{Tr}\,(D_\mu Z Z^{J-1}) \qquad \text{for } \mu = 1, \ldots, 4, \tag{3.50}$$

$$\theta_{0A}^\dagger|0, p^+\rangle \quad \hat{=} \quad \frac{1}{\sqrt{N^J}}\mathrm{Tr}\,(\psi_A Z^J) \qquad \text{for } A = 1, \ldots, 8 \tag{3.51}$$

where $\theta_{0A}^\dagger$ is the fermionic zero mode creation operator, which we have not explicitly mentioned before as we focused on the bosonic sector above. Now, we can see what the operators corresponding to the higher string excitations look like. For example, for $\Delta - J = 2$ the energy is $E_{\mathrm{lc}} = 2\mu$ and one possible state/operator pairing is

$$a_0^{\dagger \mu}a_0^{\dagger \nu}|0, p^+\rangle \,\hat{=}\, \frac{1}{\sqrt{(J-2)N^J}} \sum_{l=0}^{J-2} \mathrm{Tr}\,[D_\mu Z Z^l \mathcal{D}_\nu Z Z^{J-2-l}]. \tag{3.52}$$

Next, let us consider the non-protected first stringy states

$$a^i_{-n} \tilde{a}^j_{-n} |0, p^+\rangle, \tag{3.53}$$

which fulfil the physical state condition (3.38). The corresponding operators are called $\mathcal{O}^{ij}_n$ and should have

$$\Delta_{\mathcal{O}^{ij}_n} - J = \frac{E_{\text{lc}}}{\mu} = 2\sqrt{1 + \frac{n^2}{(\alpha' p^+ \mu)^2}}. \tag{3.54}$$

In order to check this we must translate the string parameters α', p^+, μ to gauge theory parameters. First, we use equation (3.14),

$$2p^+ = \frac{E + J}{\mu L^2} \sim \frac{2J}{\mu L^2} \quad \Rightarrow \quad (\mu p^+)^2 = \frac{J^2}{L^4}. \tag{3.55}$$

Next, we make use of $L^4 = g^2_{YM} N \alpha'^2$, which leads to

$$\frac{1}{(\alpha' p^+ \mu)^2} = \frac{L^4}{J^2 \alpha'^2} = \frac{g_{YM}^2 N}{J^2} \equiv \lambda'. \tag{3.56}$$

Hence, to first order, we obtain

$$\frac{E_{\text{lc}}}{\mu} = 2\sqrt{1 + n^2 \lambda'} \approx 2 + n^2 \lambda' + \mathcal{O}(\lambda') \tag{3.57}$$

as a function of the newly introduced quantum correction counting parameter λ', which is finite in the BMN limit. Interestingly, we see in equation (3.56) that perturbation theory around $\lambda' = 0$ is exactly the opposite limiting case of the Minkowski limit $\mu \to 0$, since $\mu \to \infty \Leftrightarrow \lambda' \to 0$.

Now, we have to find the operators $\mathcal{O}^{ij}_n$ and calculate their dimensions on the gauge theory side and see if we get the same result as in equation (3.57). Berenstein, Maldacena and Nastase managed to show that that the relevant operators are [16]

$$\mathcal{O}^{ij}_n = \frac{1}{\sqrt{JN^J}} \sum_{l=0}^{J} \text{Tr}\,[\phi_i Z^l \phi_l Z^{J-l}] e^{2\pi i n l / J} \tag{3.58}$$

with

$$\Delta_{\mathcal{O}^{ij}_n} = J + 2 + \frac{g^2_{YM} N}{J^2} n^2 + \mathcal{O}(g^4_{YM}) = J + 2 + \lambda' n^2 + \mathcal{O}(\lambda'^2). \tag{3.59}$$

The two results obtained from calculations on both sides of the duality yield the same result to order λ'. In order to check if we get the same result to all orders in gauge theory on the one hand and, on the other hand, in equation (3.57) for the

plane-wave string spectrum, we have to sum diagrams of the full planar perturbation expansion on the gauge theory side. This is necessary if the parameter λ' is not small and the expansion in equation (3.57) is not appropriate. This resummation of graphs beyond the planar limit was done for two loops in [35] and then, under certain assumptions, to all orders in [62]. This resummation confirmed the square root in equation (3.57) from the gauge theory side. We conclude that the evidence seems to strongly support that the scaling dimensions in BMN gauge theory in the planar sector reproduce the energy spectrum of the free plane-wave string.

3.6 Non-planarity in BMN gauge theory

In this last section of the chapter, we consider a specific operator as an example to show how graphs of all genera contribute to the resummation from the previous section, i.e. that we have to go beyond the planar limit to find the full expressions to all orders in the effective genus counting parameter. We stress again that the BMN limit is a new type of scaling limit and, in particular, it is non-t'Hooftian. Graphs of all genera survive when we take $N \sim J^2 \to \infty$, which was shown in [47, 22]. Non-planar graphs are still suppressed by factors of $1/N$. However, this is balanced by the growing combinatorics of diagrams involved if $J \to \infty$ at the same time. We will see an explicit example of this balancing shortly. In addition to the parameter λ' controlling the quantum expansion, there is a second parameter arising, the tuneable quotient

$$g_2 \equiv \frac{J^2}{N},\tag{3.60}$$

which controls the effective genus expansion. Therefore, we conclude that BMN gauge theory is governed by two independent parameters, λ' and g_2.

In order to show that non-planar diagrams contribute to the scaling dimension in the BMN limit and to see how g_2 emerges, we consider an explicit example. We compute the two point function

$$\langle \operatorname{Tr} Z^J(x) \operatorname{Tr} \overline{Z}^J(0) \rangle,\tag{3.61}$$

which is the simplest two point correlator in BMN gauge theory. From the action of $\mathcal{N} = 4$ SYM (2.32), we can read off the scalar propagator

$$\langle (\phi_i)_{ab}(x)(\phi_j)_{cd}(y) \rangle_0 = \frac{g_{YM}^2 \delta^{ij}}{8\pi^2 (x-y)^2} \delta_{ad} \delta_{bc}.\tag{3.62}$$

So, we can effectively reduce the 2-pt. function (3.61) to the correlator of a Gaussian

matrix model

$$\langle \operatorname{Tr} Z^J(x) \operatorname{Tr} \overline{Z}^J(0) \rangle = \left(\frac{g_{YM}^2}{8\pi^2 |x|^2} \right)^J \int dZ d\overline{Z} \operatorname{Tr} Z^J \operatorname{Tr} \overline{Z}^J e^{-\operatorname{Tr} [Z\overline{Z}]} \tag{3.63}$$

where the measure is given by

$$dZ d\overline{Z} = \prod_{a,b=1}^{N} \frac{d \operatorname{Re} Z_{ab} d \operatorname{Im} Z_{ab}}{\pi} \tag{3.64}$$

and the matrix model correlator is

$$\langle \mathcal{O} \rangle_{\mathrm{MM}} \equiv \int dZ d\overline{Z} \, \mathcal{O} \, e^{\operatorname{Tr} [Z\overline{Z}]}. \tag{3.65}$$

The propagator is given by

$$\langle Z_{ab} \overline{Z}_{cd} \rangle_{\mathrm{MM}} = \delta_{ad} \delta_{bc}. \tag{3.66}$$

This is essentially the matrix model we know from section 2.1.1.

The correlator of interest $\langle \operatorname{Tr} Z^J \overline{Z}^J \rangle_{\mathrm{MM}}$ can be computed for finite N. This was done in [47] and we quote the result here which is

$$\begin{aligned}
\langle \operatorname{Tr} Z^J \operatorname{Tr} \overline{Z}^J \rangle_{\mathrm{MM}} = J N^J \Bigg\{ 1 + \left[\binom{J}{4} + \binom{J}{3} \right] \frac{1}{N^2} \\
+ \left[21 \binom{J}{8} + 49 \binom{J}{7} + 36 \binom{J}{6} + 8 \binom{J}{5} \right] \frac{1}{N^4} + \dots \Bigg\}.
\end{aligned} \tag{3.67}$$

To find this is foremost a combinatorical problem: Take two necklaces with J beads each, one with white and one with black beads. Now, connect white and black beads with lines pair wise. The genus of a diagram is the genus of the surface with minimal genus on which the diagram can be drawn without crossing lines. Then, we notice that in the $N, J \to \infty$, $J^2/N = \mathrm{const.}$ limit the correlator becomes

$$\frac{1}{J N^J} \langle \operatorname{Tr} Z^J \operatorname{Tr} \overline{Z}^J \rangle_{\mathrm{MM}} \to 1 + \frac{1}{24} \frac{J^4}{N^2} + \frac{21}{8!} \frac{J^8}{N^4} + \dots \overset{[47]}{=} \frac{2N}{J^2} \sinh \left(\frac{J^2}{2N} \right). \tag{3.68}$$

We see that in the BMN limit the genus counting parameter $g_2 = J^2/N$ appears. In each summand only the term with highest power in J survives. Hence, only a small amount of the original non-planar graphs survive the limit. However, at every order (genus) there is a diagram contributing to the correlator. Therefore, the BMN limit is non-planar. Even the lowest order terms in λ' contain diagrams that are highly non-planar, i.e. which have a high genus.

We have seen in section 2.5 that Maldacena originally proposed the correspondence to hold in the planar limit only and that today, however, we believe it to hold more generally. Since graphs of all genera contribute in the BMN limit, we have found a possibility to test the correspondence beyond the planar limit. If we find that the resummation of graphs of all topologies on the gauge theory side yields the right expressions for the scaling dimensions to all powers in J^2/N, then this confirms the stronger formulation of the AdS/CFT correspondence.

4 String solutions with large spins and R-charges

As we have seen in chapter 3, Berenstein, Maldacena and Nastase [16] found a sub-sector of string states and the corresponding gauge theory operators enabling them to test the AdS/CFT conjecture beyond the supergravity limit. Inspired by their success, string states with high R-charges were studied extensively in the following years. The BMN string is a very simplified model describing a pointlike string, stationary in AdS_5 and only rotating on a geodesic in S^5. The model can be generalised in many ways. Gubser, Klebanov and Polyakov studied closed, folded strings, which are extended and spinning in the AdS_5 component of $AdS_5 \times S^5$. They also discussed extended, folded strings spinning in one plane of the S^5 component [37]. We will discuss these classical solutions of the superstring σ-model in $AdS_5 \times S^5$ in section 4.1.

Frolov and Tseytlin made several contributions to the field. They studied classical multi-spin solutions and derived the corresponding energies in terms of the BMN effective coupling λ/J^2 in the BMN limit. Partly, they confirmed results that had already been obtained on the gauge theory side [9]. They also introduced a semi-classical quantization procedure for rotating strings in $AdS_5 \times S^5$ [28, 30]. In section 4.2, we will study the folded and circular string rotating in two orthogonal planes of the S^5, which were derived in [29, 31].

4.1 The GKP string

After Berenstein, Maldacena and Nastase made their discovery [16] that high R-charge strings effectively live in a simple plane-wave background, the Penrose limit of $AdS_5 \times S^5$, and thus could be quantized, it remained unclear if this success was a hint of a deeper general mechanism that works for all high spin or high R-charge strings. Gubser, Klebanov and Polyakov found that large spin operators correspond to certain solitonic solutions of the $AdS_5 \times S^5$ σ-model. We will study their derivation [37] in detail and, thus, see another useful application of considering states with large charges.

4.1.1 Strings with large spin in AdS_5

Let us first consider a string only rotating in AdS_5. This is different from the rotations in S^5 that were considered in the BMN approach. In particular, spinning in the AdS_5 component is not connected to supersymmetry in the dual gauge theory model. We use the global AdS_5 metric (3.2),

$$\begin{aligned}
ds^2_{AdS_5} &= L^2(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3{}^2) \\
&= L^2(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, (d\overline{\gamma}^2 + \cos^2 \overline{\gamma} \, d\varphi_1^2 + \sin^2 \gamma \, d\varphi_2^2)), \quad (4.1)
\end{aligned}$$

and consider a string that is stretched in the ρ-direction with the maximum radial coordinate ρ_0. Moreover, the string is rotating on the equator of S^3, i.e. the trajectory is given by

$$t = e\tau, \quad \rho = \rho(\sigma), \quad \overline{\gamma} = 0, \quad \varphi_1 = e\omega\tau. \tag{4.2}$$

The Lagrangian $\mathcal{L}$ (not the Lagrangian density) in the Nambu-Goto action is

$$\mathcal{L} = -\frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \sqrt{-h} \tag{4.3}$$

where $h = \det(h_{ab})$ and $h_{ab} = \partial_a y^m \partial_b y^n g^{(AdS_5)}_{mn}$ is the pull-back of the spactime metric to the worldsheet, $a, b \in \{\sigma, \tau\}$, $y^m = (t, \rho, \overline{\gamma}, \varphi_1, \varphi_2)$. This yields

$$\begin{aligned}
\mathcal{L} &= -\frac{L^2}{2\pi\alpha'} \int_0^{2\pi} d\sigma \sqrt{(\cosh^2 \rho \, \dot{t}^2 - \sinh^2 \rho \, \dot{\varphi}_1^2)\rho'^2} \\
&= -\frac{L^2}{2\pi\alpha'} \int_0^{2\pi} d\sigma \frac{d\rho}{d\sigma} \sqrt{\cosh^2 \rho \, \dot{t}^2 - \sinh^2 \rho \, \dot{\varphi}_1^2} \\
&= -\frac{4L^2}{2\pi\alpha'} \int_0^{\rho_0} d\rho \sqrt{\cosh^2 \rho \, \dot{t}^2 - \sinh^2 \rho \, \dot{\varphi}_1^2} \tag{4.4}
\end{aligned}$$

where the factor 4 appears since the segment from $\rho = 0$ to $\rho = \rho_0$ is integrated four times when we integrate around the folded string once. The maximum radial coordinate ρ_0 is given by

$$\cosh^2 \rho_0 = \omega^2 \sinh^2 \rho_0$$
$$\Leftrightarrow \coth^2 \rho_0 = \omega^2. \tag{4.5}$$

The conserved charges corresponding to the two cyclic coordinates t and φ_1 are the energy E and spin S,

$$E = -\frac{\partial L}{\partial \dot{t}} = \frac{4L^2}{2\pi\alpha'} \int_0^{\rho_0} d\rho \, \frac{\cosh^2 \rho \, \dot{t}}{\sqrt{\cosh^2 \rho \, \dot{t}^2 - \sinh^2 \rho \, \dot{\varphi}_1^2}},$$

$$S = \frac{\partial L}{\partial \dot\varphi_1} = \frac{4L^2}{2\pi\alpha'} \int_0^{\rho_0} d\rho \, \frac{\sinh^2 \rho \, \dot\varphi_1}{\sqrt{\cosh^2 \rho \, \dot t^2 - \sinh^2 \rho \, \dot\varphi_1^2}}. \tag{4.6}$$

For the explicit parametrisation we introduced in equation (4.2), we get

$$\mathcal{L} = -\frac{4L^2}{2\pi\alpha'} \int_0^{\rho_0} d\rho \sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho},$$

$$E = \frac{4L^2}{2\pi\alpha'} \int_0^{\rho_0} d\rho \, \frac{\cosh^2 \rho}{\sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}},$$

$$S = \frac{4L^2}{2\pi\alpha'} \int_0^{\rho_0} d\rho \, \frac{\omega \sinh^2 \rho}{\sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}}. \tag{4.7}$$

Now, considering the case of large ω, we get

$$\tanh \rho_0 = \frac{1}{\omega} \ll 1 \qquad \Rightarrow \qquad \rho_0 \approx \frac{1}{\omega} \tag{4.8}$$

from equation (4.5). Consequently, the string is not stretched a lot along the radial coordinate of AdS_5. Therefore, we can approximate the metric for $\rho \approx 0$ since the string is located in the centre of AdS_5. In this limit, the metric in global coordinates reduces to

$$ds^2_{AdS_5} = L^2(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3{}^2)$$

$$\approx L^2(-dt^2 d\rho^2 + \rho^2 d\Omega_3{}^2), \tag{4.9}$$

which is simply flat space. This yields the Lagrangian

$$\mathcal{L} = -\frac{4L^2}{2\pi\alpha'} \int_0^{1/\omega} d\rho \sqrt{\dot t^2 - \rho^2 \dot\varphi_1^2}. \tag{4.10}$$

According to equation (4.7), the conserved charges are

$$\begin{aligned}
E &= \frac{4L^2}{2\pi\alpha'} \int_0^{1/\omega} d\rho \frac{1}{\sqrt{1 - \rho^2\omega^2}} \\
&= \frac{4L^2}{2\pi\alpha'} \int_0^{\pi/2} \frac{dx}{\omega} \frac{\cos x}{\sqrt{1 - \sin^2 x}} \qquad \text{(substituted } \omega\rho = \sin x\text{)} \\
&= \frac{L^2}{\alpha'\omega} = \sqrt{\lambda}\omega
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
S &= \frac{4L^2}{2\pi\alpha'} \int_0^{1/\omega} d\rho \frac{\omega\rho^2}{\sqrt{1 - \rho^2\omega^2}} \\
&= \frac{4L^2}{2\pi\alpha'} \int_0^{\pi/2} \frac{dx}{\omega} \frac{\cos x \sin^2 x}{\omega\sqrt{1 - \sin^2 x}} \qquad \text{(substituted } \omega\rho = \sin x\text{)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4L^2}{2\pi\alpha'\omega^2} \int_0^{\pi/2} dx \, \sin^2 x \\
&= \frac{L^4}{2\alpha'\omega^2} = \frac{\sqrt{\lambda}}{2\omega^2}
\end{aligned}
\tag{4.12}
$$

such that we can now write down the sought-after expression for the energy in terms of spin,

$$
E^2 = E^2(S) = L^2 \frac{2S}{\alpha'} = 2\sqrt{\lambda}S.
\tag{4.13}
$$

However, we see from equation (4.12) that for large ω we have $S \ll \sqrt{\lambda}$. We are more interested in the large spin limit since we want to know if the large spin limit is similar to or different from the large R-charge limit.

So, let us consider the case

$$
\omega = 1 + 2\eta
\tag{4.14}
$$

and examine the limit when $\eta \to 0$, which corresponds to $S \gg \sqrt{\lambda}$. From this expression for ω, we obtain

$$
\rho_0 = \operatorname{arcoth}(1 + 2\eta) \approx \frac{1}{2} \ln \frac{1}{\eta} + \mathcal{O}(\eta).
\tag{4.15}
$$

An expansion of the expressions for energy and spin in equation (4.7) yields [37]

$$
\begin{aligned}
E &= \frac{\sqrt{\lambda}}{2\pi} \left(\frac{1}{\eta} + \ln\left(\frac{1}{\eta}\right) + \mathcal{O}(\eta) \right), \\
S &= \frac{\sqrt{\lambda}}{2\pi} \left(\frac{1}{\eta} - \ln\left(\frac{1}{\eta}\right) + \mathcal{O}(\eta) \right), \\
\Rightarrow \quad E - S &= \frac{\sqrt{\lambda}}{\pi} \ln\left(\frac{S}{\sqrt{\lambda}}\right) + \text{const.} + \mathcal{O}\left(\frac{\sqrt{\lambda}}{S}\right)
\end{aligned}
\tag{4.16}
$$

where we have used

$$
\ln\left(\frac{1}{\eta}\right) + \mathcal{O}(\eta) = \ln\left(\frac{2\pi S}{\sqrt{\lambda}}\right) = \ln\left(\frac{S}{\sqrt{\lambda}}\right) + \text{const.}
\tag{4.17}
$$

and

$$
\mathcal{O}(\eta) = \mathcal{O}\left(\frac{\sqrt{\lambda}}{S}\right).
\tag{4.18}
$$

So, the anomalous dimension grows logarithmically with the spin S. In what follows, we want to contrast this with the extended, folded string rotating in the S^5 component, which we examine in section 4.1.2.

4.1.2 String with large spin in S^5

Consider a string whose centre of mass is not moving in S^5. However, we assume it is rotating and, therefore, it is correspondingly being stretched. The centre of mass is assumed to be located at the north pole of the S^5 with the metric

$$
\begin{aligned}
ds_{S^5}^2 &= L^2(\cos^2\theta \, d\psi^2 + d\theta^2 + \sin^2\theta \, d\Omega_3'^2) \\
&= L^2(\cos^2\theta \, d\psi^2 + d\theta^2 + \sin^2\theta(d\gamma^2 + \cos^2\gamma \, d\phi_1^2 + \sin^2\gamma \, d\phi_2^2)). \quad (4.19)
\end{aligned}
$$

We use the following ansatz for the trajectory:

$$
t = e\tau, \quad \rho = 0, \quad \gamma = e\omega\tau, \quad \theta = \theta(\sigma) \quad (4.20)
$$

while the string is stationary in AdS_5 and the other angles in S^5 are constant. The Virasoro constraints are

$$
\begin{aligned}
0 &= \dot{y}^m \dot{y}_m + y'^m y'_m + \dot{x}^m \dot{x}_m + x'^m x'_m, \\
0 &= \dot{y}^m y'_m + \dot{x}^m x'_m \quad (4.21)
\end{aligned}
$$

as we will derive later in equation (4.35). The y^m are contracted with $g_{mn}^{(AdS_5)}(y)$, the x^m with $g_{mn}^{(S^5)}(x)$. The first constraint yields

$$
\begin{aligned}
0 &= -\dot{t}^2 + \theta'^2 - \sin^2\theta \, \dot{\gamma}^2 \\
\Rightarrow \theta'^2 &= e^2(1 - \omega^2 \sin^2\theta) \\
\Rightarrow \frac{d\theta}{d\sigma} &= e\sqrt{1 - \omega^2 \sin^2\theta} \quad (4.22)
\end{aligned}
$$

while the second one is identically fulfilled. The Lagrangian of the Polyakov action in conformal gauge is

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{4\pi\alpha'} \int_0^{2\pi} d\sigma \, (\partial_a y^m \partial^a y_m + \partial_b x^m \partial^a x_m) \\
&= \frac{L^2}{4\pi\alpha'} \int_0^{2\pi} d\sigma \, (-\dot{t}^2 + \sin^2\theta \, \dot{\gamma}^2 + \theta'^2) \quad (4.23)
\end{aligned}
$$

Consequently, the energy E and the R-charge are the canonically conjugate momenta of t and γ,

$$
\begin{aligned}
E &= -\frac{\partial L}{\partial \dot{t}} = \frac{L^2}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, \dot{t} \\
&= \frac{L^2}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, e = \frac{4L^2}{2\pi\alpha'} \int_0^{\theta_0} d\theta \, \frac{1}{\sqrt{1 - \omega^2 \sin^2\theta}},
\end{aligned}
$$

$$J = \frac{\partial L}{\partial \dot{\gamma}} = \frac{L^2}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, \sin^2\theta \, \dot{\gamma}$$
$$= \frac{L^2}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, e\omega \sin^2\theta = \frac{4\omega L^2}{2\pi\alpha'} \int_0^{\gamma_0} d\gamma \, \frac{\sin^2\theta}{\sqrt{1-\omega^2\sin^2\theta}} \qquad (4.24)$$

where we used equation (4.22) to change the integration variable from σ to γ. γ_0 is the maximum value of γ before the string folds back onto itself and is given by

$$\sin\gamma_0 = \frac{1}{\omega}. \qquad (4.25)$$

Again, choosing $\omega = 1 + 2\eta$ and considering the limit $\eta \to 0 \Rightarrow \theta_0 = \pi/2$ we find [37]

$$E - J \to \frac{2L^2}{\pi\alpha'} = \frac{2\sqrt{\lambda}}{\pi}. \qquad (4.26)$$

So, in contrast to $E - S \sim \sqrt{\lambda} \ln S$ for the extended, folded string rotating in AdS_5, we find $E - J \sim \sqrt{\lambda}$ for the string rotating in the S^5 component. We will generalise and deepen our treatment of the string spinning in S^5 in the next section.

4.1.3 Gauge theory implications

The result from equation (4.16) translated to gauge theory means that

$$\Delta - S = \frac{\sqrt{\lambda}}{\pi} \ln\left(\frac{S}{\sqrt{\lambda}}\right) + \mathcal{O}(S^0), \qquad (4.27)$$

i.e. the anomalous dimensions Δ of the operators corresponding to the strings spinning in AdS_5 grow logarithmically with the spin S. This is a result known from gauge theories, even for non-supersymmetric ones, for which it was argued that anomalous dimensions scale like $\ln S$ to all orders both perturbatively and non-perturbatively [43, 45]. It is exciting to see the same behaviour appearing on the string side of the AdS/CFT correspondence.

The existence of operators with large spin S, unlike those of large R-charge J, does not depend on the supersymmetry of the gauge theory. Consequently, they are also present in gauge theories, such as QCD. In $\mathcal{N} = 4$ SYM, large S operators look, for example, like

$$\mathrm{Tr}\left[\phi^i \nabla_{(\mu_1} \cdots \nabla_{\mu_S)} \phi^i\right], \qquad (4.28)$$

which has bare dimension $\Delta_0 = S + 2$. In QCD, anomalous dimensions, such as $\Delta = (S + 2) + f(\lambda) \ln S$, could be probed experimentally. Gubser, Klebanov and Polyakov have succeeded in calculating the dimensions for large spin S finding a soliton solution of the σ-model of type IIB strings in $AdS_5 \times S^5$ in a classical way. Consequently, we suspect that highly excited gauge theory operators are identified

with σ-model solitons in a more general context.

In [28], Frolov and Tseytlin perform a semi-classical quantization of superstrings in $AdS_5 \times S^5$. They start with the classical solution of a string rotating in AdS_5 and moving on a great circle of S^5, i.e. the solution has both angular momentum S and R-charge J. The energy becomes a function of the two charges. The string one-loop corrections to the classical energy are derived for small σ-model coupling $1/\sqrt{\lambda} \ll 1$ and the $E - S \sim \ln S$ behaviour in the $S \neq 0$ sector is recovered. Consequently, the findings of GKP are confirmed at one loop and the perturbative gauge theory expectation for the scaling dimension is met.

4.2 Strings spinning in the S^5 component

In this section, we generalise the GKP ansatz of a string spinning in the S^5 component of $AdS_5 \times S^5$ from section 4.1.2. We focus on the results by Frolov and Tseytlin [29, 31], which have been reviewed by Plefka [56] and Tseytlin himself in [69]. The purpose is to build up the necessary understanding of the string side large R-charge solutions in order to see the connection to spin chains in gauge theories, which we will study in chapter 5.

4.2.1 The general setting

The $AdS_5 \times S^5$ metric (3.2) is

$$
\begin{aligned}
ds^2_{Ads_5 \times S^5} &= L^2(ds^2_{AdS_5} + ds^2_{S^5}), \\
ds^2_{AdS_5} &= -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3{}^2, \\
ds^2_{S^5} &= \cos^2 \theta \, d\psi^2 + d\theta^2 + \sin^2 \theta \, d\Omega'_3{}^2, \\
d\Omega_3{}^2 &= d\overline{\gamma}^2 + \cos^2 \overline{\gamma} \, d\varphi_1^2 + \sin^2 \overline{\gamma} \, d\varphi_2^2, \\
d\Omega'_3{}^2 &= d\gamma^2 + \cos^2 \gamma \, d\phi_1^2 + \sin^2 \gamma \, d\phi_2^2.
\end{aligned}
\tag{4.29}
$$

Let us denote $y^m = (\rho, t, \overline{\gamma}, \varphi_1, \varphi_2)$ and $x^m = (\psi, \theta, \gamma, \phi_1, \phi_2)$ with $m = 1, \ldots, 5$. Then, the bosonic part of the Polyakov action in conformal gauge is

$$
S_P = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \, \eta^{ab} \left(\partial_a y^m \partial_b y^n g^{(AdS_5)}_{mn}(y) + \partial_a x^m \partial_b x^n g^{(S^5)}_{mn}(x) \right)
\tag{4.30}
$$

where we have used that $L^2/\alpha' = \sqrt{\lambda}$. Examining S_P using the metric (4.29), we observe that the Lagrangian associated with the Polyakov action has six cyclic coordinates $(t, \varphi_1, \varphi_2, \psi, \phi_1, \phi_2)$, i.e. the Lagrangian does not explicitly depend on these coordinates. We have seen these symmetries before when we derived the isometry group of $AdS_5 \times S^5$ in section 2.4.3. The conserved charges associated with

the six cyclic coordinates are the Cartan generators of $SO(2,4) \times SO(6)$ and they
are given by

$$
\begin{aligned}
E &= \frac{\partial L}{\partial \dot{t}} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2 \rho \, \dot{t}, \\
S_1 &= -\frac{\partial L}{\partial \dot{\phi}_1} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2 \rho \, \cos^2 \overline{\gamma} \, \dot{\varphi}_1, \\
S_2 &= -\frac{\partial L}{\partial \dot{\phi}_2} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2 \rho \, \sin^2 \overline{\gamma} \, \dot{\varphi}_2, \\
J_1 &= -\frac{\partial L}{\partial \dot{\phi}_1} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2 \theta \, \cos^2 \gamma \, \dot{\phi}_1, \\
J_2 &= -\frac{\partial L}{\partial \dot{\phi}_2} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2 \theta \, \sin^2 \gamma \, \dot{\phi}_1, \\
J_3 &= -\frac{\partial L}{\partial \dot{\psi}} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2 \theta \, \dot{\psi}.
\end{aligned}
\tag{4.31}
$$

The charges E, S_1 and S_2 are related to the $SO(2,4)$ factor of the symmetry while
J_1, J_2 and J_3 are associated with $SO(6)$. Therefore, S_1 and S_2 are angular momenta
while J_1, J_2 and J_3 are R-charges.

Reintroducing the worldsheet metric h_{ab} into the action before choosing the gauge
$h_{ab} = \eta_{ab}$, we can derive the Virasoro constraints, which are given by the equations
of motion for the worldsheet metric. The action before gauge fixing reads

$$
S_{\text{ungauged}} = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \sqrt{h} h^{ab} \left(\partial_a y^m \partial_b y^n g_{mn}^{(AdS_5)}(y) + \partial_a x^m \partial_b x^n g_{mn}^{(S^5)}(x) \right)
\tag{4.32}
$$

where $h = |\det h|$. Now, use that

$$
\delta \sqrt{h} = -\frac{\sqrt{h}}{2} h_{ab} \delta h^{ab}
\tag{4.33}
$$

to conclude that the Virasoro constraints are given by

$$
\begin{aligned}
0 &= \frac{\delta S_{\text{ungauged}}}{\delta h^{ab}} \\
&= -\frac{\sqrt{h}}{2} h_{ab} h^{cd} (\partial_c y^m \partial_d y_m + \partial_c x^m \partial_d x_m) + \sqrt{h} (\partial_a y^m \partial_b y_m + \partial_a x^m \partial_b x_m)
\end{aligned}
\tag{4.34}
$$

where the x^m and y^m are contracted with the appropriate metric $g_{mn}^{(AdS_5)}$ and $g_{mn}^{(S^5)}$,
respectively. Upon gauge fixing, $h_{ab} = \eta_{ab} \Rightarrow h = 1$, this yields two independent
equations,

$$
\begin{aligned}
0 &= \dot{y}^m \dot{y}_m + \dot{x}^m \dot{x}_m + y'^m y'_m + x'^m x'_m, \\
0 &= \dot{y}^m y'_m + \dot{x}^m x'_m.
\end{aligned}
\tag{4.35}
$$
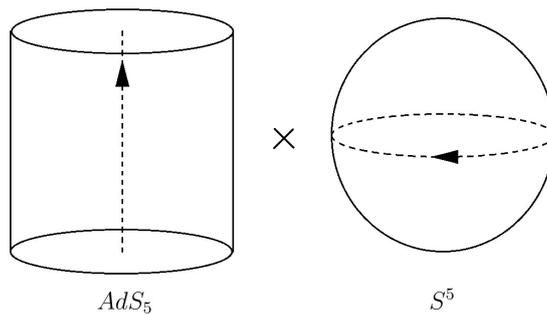
Figure 4.1: Trajectory of a pointlike string stationary in the centre of $AdS_5 \times S^5$ and moving on a great circle of S^5.

The Virasoro constraints have to be implemented in addition to the equations of motion, which are given by

$$\partial_a \left(g_{mn}^{(AdS_5)}(y) \partial^a y^n \right) = 0 \qquad \text{and} \qquad \partial_a \left(g_{mn}^{(S^5)}(x) \partial^a x^n \right) = 0. \qquad (4.36)$$

We conclude that, even though $AdS_5 \times S_5$ is highly symmetric, string theory in this space is very non-trivial. The equations of motion are non-linear and therefore very hard to quantize (as we cannot just expand solutions in plane waves and use the usual canonical quantization). This prevents us from directly checking the conjectured duality to $\mathcal{N} = 4$ SYM.

From the AdS/CFT correspondence, we know that string solutions in $AdS_5 \times S^5$ correspond to operators in $\mathcal{N} = 4$ SYM with gauge group SU(N). The relation between string state energies and SYM operator scaling dimensions is given by equation (2.136),

$$\Delta \left(\lambda, \frac{1}{N} \right) = E \left(\frac{L^2}{\alpha'}, g_s \right). \qquad (4.37)$$

We derive several string solutions beyond the pointlike BMN ansatz and express their energies E in terms of the other conserved charges, $E = E(S_1, S_2, J_1, J_2, J_3)$. Then, we will try to identify these energies with gauge theory operators with an identical set of charges $(S_1, S_2, J_1, J_2, J_3)$.

4.2.2 Classical BMN solution revisited

We briefly repeat the classical solution from section 3.1 of a pointlike string stationary in AdS_5 and rotating on a great circle in S^5. The symmetries of S^5 allow us to choose this great circle to be the equator. The trajectory is displayed in figure 4.1. In contrast to earlier, we introduce another parameter κ to parametrize the solution. We will see its dependence on the parameters from section 3.1, J and λ, later. The

rotating point particle on S^5 is given by

$$t = \kappa\tau, \quad \rho = 0, \quad \theta = 0, \quad \psi = \kappa\tau \tag{4.38}$$

with the other coordinates vanishing. We can now compute the conserved charges from equation (4.31) to be

$$E = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2\rho\, \dot{t} = \sqrt{\lambda}\kappa,$$

$$J_3 = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2\theta\, \dot{\psi} = \sqrt{\lambda}\kappa \tag{4.39}$$

while the other charges vanish. Hence, the angular momentum on the sphere, J, is

$$J = J_3 = \sqrt{\lambda}\kappa \qquad \Rightarrow \qquad \kappa = \frac{J}{\sqrt{\lambda}}, \tag{4.40}$$

which is exactly the parameter held constant in the BMN limit, $J \to \infty$ and $\lambda \to \infty$ but J^2/λ constant.

4.2.3 Spinning string solutions

Let us consider an extended string stationary in AdS_5 and rotating in two orthogonal planes if we think of the S^5 being embedded in $\mathbb{R}^6$, i.e. the string rotates within an $S^3 \subset S^5$. This has first been discussed by Frolov and Tseytlin [31, 29], whose treatment we will follow in this section. The trajectory of the string is given by

$$t = \kappa\tau, \quad \rho = 0, \quad \theta = \frac{\pi}{2}, \quad \psi = 0, \quad \gamma = \gamma(\sigma), \quad \phi_1 = \omega_1\tau, \quad \phi_2 = \omega_2\tau. \tag{4.41}$$

Then, the bosonic part of the Polyakov action (4.30) is

$$S_P = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left(\dot{t}^2 - \cos^2\gamma\, \dot{\phi_1}^2 - \sin^2\gamma\, \dot{\phi_2}^2 + \gamma'^2 \right)$$

$$= -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left(\kappa^2 - \cos^2\gamma\, \omega_1^2 - \sin^2\gamma\, \omega_2^2 + \gamma'^2 \right) \tag{4.42}$$

The integrand is now independent of τ. In consequence, we have a one parameter problem, which yields an equation of motion for γ,

$$0 = \frac{d}{d\sigma}\frac{\partial L}{\partial\gamma'} - \frac{\partial L}{\partial\gamma}$$

$$= -\frac{\sqrt{\lambda}}{2\pi}\left(\gamma'' - \cos\gamma\,\sin\gamma\,(\omega_1^2 - \omega_2^2) \right)$$

$$\Rightarrow \quad 0 = \gamma'' + \omega_{21}^2 \cos\gamma\,\sin\gamma \tag{4.43}$$

where we have defined $\omega_{21}^2 = \omega_2^2 - \omega_1^2$. We integrate equation (4.43) using an integrating factor,

$$\gamma'' = -\omega_{21}^2 \cos\gamma \sin\gamma$$

$$\Rightarrow \qquad \gamma''\gamma' = -\omega_{21}^2 \cos\gamma \sin\gamma \, \gamma'$$

$$\Rightarrow \qquad \frac{d}{d\sigma}(\gamma')^2 = -\omega_{21}^2 \frac{d}{d\sigma} \sin^2\gamma$$

$$\Rightarrow \qquad \left(\frac{d\gamma}{d\sigma}\right)^2 - \left(\frac{d\gamma}{d\sigma}\bigg|_{\sigma=0}\right)^2 = \omega_{21}^2 \left(\sin^2\gamma\big|_{\sigma=0} - \sin^2\gamma\right)$$

$$\Rightarrow \qquad \frac{d\gamma}{d\sigma} = \omega_{21}\sqrt{q - \sin^2\gamma} \qquad (4.44)$$

where

$$q = \sin^2\gamma\big|_{\sigma=0} + \frac{1}{\omega_{21}^2}\left(\frac{d\gamma}{d\sigma}\bigg|_{\sigma=0}\right)^2 \qquad (4.45)$$

is a constant of integration. The first of the Virasoro constraints (4.35) amounts to

$$0 = \dot{y}^m \dot{y}_m + \dot{x}^m \dot{x}_m + y'^m y'_m + x'^m x'_m$$

$$= -\kappa^2 + \omega_1^2 \cos^2\gamma + \omega_2^2 \sin^2\gamma + \gamma'^2$$

$$\overset{(4.2.3)}{=} -\kappa^2 + \omega_1^2 \cos^2\gamma + \omega_2^2 \sin^2\gamma + \omega_{21}^2 \left(q - \sin^2\gamma\right)$$

$$= -\kappa^2 + \omega_1^2 + \omega_{21}^2 q$$

$$\Rightarrow \quad q = \frac{\kappa^2 - \omega_1^2}{\omega_{21}^2} \qquad (4.46)$$

while the second constraint gives no further restriction. The three non-zero charges are

$$E = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2\rho \, \dot{t}^2 = \sqrt{\lambda}\kappa,$$

$$J_1 = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2\theta \, \cos^2\gamma \, \dot{\phi}_1 = \sqrt{\lambda}\omega_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2\gamma,$$

$$J_2 = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2\theta \, \sin^2\gamma \, \dot{\phi}_2 = \sqrt{\lambda}\omega_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2\gamma. \qquad (4.47)$$

Therefore, we can conclude that

$$\sqrt{\lambda} = \frac{J_1}{\omega_1} + \frac{J_2}{\omega_2}. \qquad (4.48)$$

In the following, we consider three separate cases. First, we consider a folded string associated with $q < 1$. In the second, trivial case for $\omega_1 = \omega_2$, we obtain a circular string with equal charges J_1 and J_2. Finally, we study circular strings associated with the solution for $q > 1$.

The folded string

If the spinning, closed string is folded, it reaches some maximum angle γ_0 before it folds back onto itself. Let us choose the coordinate σ such that $\sigma = 0$ is at one of the two points where the string folds. In consequence, we get an explicit expression for the integration constant q:

$$\gamma|_{\sigma=0} = \sigma_0, \qquad \frac{d\gamma}{d\sigma}\bigg|_{\sigma=0} = 0 \qquad \overset{(4.45)}{\Rightarrow} \qquad q = \sin^2\gamma_0. \qquad (4.49)$$

From this expression, it is obvious that the folded string corresponds to the case $q < 1$.

We introduce two elliptic integrals that will appear in our further discussions,

$$E(x) = \int_0^{\pi/2} d\gamma \sqrt{1 - x\sin^2\gamma} \qquad \text{and} \qquad K(x) = \int_0^{\pi/2} d\gamma \frac{1}{\sqrt{1 - x\sin^2\gamma}}. \qquad (4.50)$$

Now, let us find a simplified expression for J_1. Since the string folds back onto itself, we only have to integrate over σ such that $0 \leq \gamma \leq \gamma_0$ is only covered once, i.e.

$$\int_0^{2\pi} d\sigma\,(\ldots) = 4\int_0^{\gamma_0} d\gamma \frac{d\sigma}{d\gamma}\,(\ldots) = 4\int_0^{\gamma_0} \frac{d\gamma}{\omega_{21}\sqrt{\sin^2\gamma_0 - \sin^2\gamma}}(\ldots) \qquad (4.51)$$

where we have made use of equation (4.2.3). Now, we can derive an expression for the R-charge J_1,

$$
\begin{aligned}
J_1 &= \frac{\sqrt{\lambda}\omega_1}{2\pi} \int_0^{2\pi} \frac{d\sigma}{2\pi}\cos^2\psi \\
&= \frac{4\sqrt{\lambda}\omega_1}{2\pi} \int_0^{\gamma_0} d\gamma \frac{\cos^2\gamma}{\omega_{21}\sqrt{\sin^2\gamma_0 - \sin^2\gamma}} \\
&\quad (\text{substitute } \sin\gamma = \sin\gamma_0\sin\chi \Rightarrow \cos\gamma\,d\gamma = \sin\gamma_0\cos\chi\,d\chi) \\
&= \frac{2\sqrt{\lambda}\omega_1}{\pi\omega_{21}} \int_0^{\pi/2} \underbrace{d\chi\,\sin\gamma_0\cos\chi}_{=d\gamma\,\cos\gamma} \underbrace{\sqrt{1 - \sin^2\gamma_0\sin^2\chi}}_{=\cos\chi} \frac{1}{\sqrt{\sin^2\gamma_0 - \sin^2\gamma_0\sin^2\chi}} \\
&= \frac{2\sqrt{\lambda}\omega_1}{\pi\omega_{21}} \int_0^{\pi/2} d\chi\,\sqrt{1 - \sin^2\gamma_0\sin^2\chi} \\
&= \frac{2\sqrt{\lambda}\omega_1}{\pi\omega_{21}} E(\sin^2\gamma_0). \qquad (4.52)
\end{aligned}
$$

Furthermore, we get another relation with the help of the same substitution

$$2\pi = \int_0^{2\pi} d\sigma = 4\int_0^{\gamma_0} \frac{d\gamma}{\omega_{21}\sqrt{\sin^2\gamma_0 - \sin^2\gamma}}$$

$$= \frac{4}{\omega_{21}} \int_0^{\pi/2} d\chi \, \frac{\sin\gamma_0 \, \cos\chi}{\sqrt{1 - \sin^2\gamma_0 \sin^2\chi}} \frac{1}{\sqrt{\sin^2\gamma_0 - \sin^2\gamma_0 \sin^2\chi}}$$

$$= \frac{4}{\omega_{21}} \int_0^{\pi/2} \frac{d\chi}{\sqrt{1 - \sin^2\gamma_0 \sin^2\chi}}$$

$$= \frac{4}{\omega_{21}} K(\sin^2\gamma_0) \tag{4.53}$$

Let us put everything together. From equation (4.31) we get $\kappa = E/\sqrt{\lambda}$ and we can therefore write equations (4.52) and (4.53) as

$$\frac{\omega_1}{\omega_{21}} = \frac{\pi}{2\sqrt{\lambda}} \frac{J_1}{E(\sin^2\gamma_0)}, \qquad\qquad \frac{\kappa}{\omega_{21}} = \frac{\pi}{2\sqrt{\lambda}} \frac{E}{K(\sin^2\gamma_0)}. \tag{4.54}$$

Using equation (4.48) we get

$$\frac{J_1}{\omega_1} = \sqrt{\lambda} - \frac{J_2}{\omega_2} \tag{4.55}$$

which yields

$$K(q) - E(q) = \frac{\pi\omega_{21}}{2}\left(1 - \frac{J_1}{\sqrt{\lambda}\omega_1}\right) = \frac{\pi\omega_{21}}{2}\left(1 - \frac{1}{\sqrt{\lambda}}\left(\sqrt{\lambda} - \frac{J_2}{\omega_2}\right)\right)$$

$$= \frac{\pi J_2}{2\sqrt{\lambda}} \frac{\omega_{21}}{\omega_2}$$

$$\Rightarrow \quad \frac{\omega_2}{\omega_{21}} = \frac{\pi}{2\sqrt{\lambda}} \frac{J_2}{K(q) - E(q)}. \tag{4.56}$$

Finally, the Virasoro constraint 4.46 gives us

$$q^2 = \frac{\kappa^2 - \omega_1^2}{\omega_{21}^2} = \frac{\pi^2}{4\lambda} \frac{E^2}{K(q)^2} - \frac{\pi^2}{4\lambda} \frac{J_1^2}{E(q)^2}$$

$$\Rightarrow \quad \frac{4q\pi}{\pi^2} = \frac{E^2}{K(q)^2} - \frac{J_1^2}{E(q)^2} \tag{4.57}$$

and

$$1 = \frac{\omega_2^2 - \omega_1^2}{\omega_{21}^2} = \frac{\pi^2}{4\lambda} \frac{J_2^2}{(K(q) - E(q))^2} - \frac{\pi^2}{4\pi} \frac{J_1^2}{E(q)^2}$$

$$\Rightarrow \quad \frac{4\lambda}{\pi^2} = \frac{J_2^2}{(K(q) - E(q))^2} - \frac{J_1^2}{E(q)}^2 \tag{4.58}$$

The equations (4.57) and (4.58) implicitly define the energy in terms of the other charges, $E = E(J_1, J_2)$, upon elimination of q.

Naturally, we are interested in the perturbative expansion of the energy with parameter λ/J^2 in the BMN limit $J = J_1 + J_2 \to \infty$. In this limit, the results can be compared to gauge theory calculations, i.e. the energies can be compared to scaling dimensions of the corresponding operators with the same R-charges J. We

assume that q and E have an analytic behaviour in λ/J^2, which means that we can expand them in power series,

$$q = q_0 + \frac{\lambda}{J^2}q_1 + \frac{\lambda^2}{J^4}q_2 + \dots,$$

$$E = J\left(E_0 + \frac{\lambda}{J^2}E_1 + \frac{\lambda^2}{J^4}E_2 + \dots\right). \tag{4.59}$$

Plugging these expansions into the equations (4.57) and (4.58) and solving order by order in λ/J^2 yields $E_0 = 1$ and q_0 being defined by [56]

$$\frac{J_2}{J} = 1 - \frac{E(q_0)}{K(q_0)}. \tag{4.60}$$

The first non-trivial term in the energy is

$$E_1 = \frac{2}{\pi^2}K(q_0)\left(E(q_0) - (1 - q_0)K(q_0)\right). \tag{4.61}$$

In the case $J_1 = J_2$, Frolov and Tseytlin derived [31]

$$E(J_1, J_2 = J_1) = 2J_1\left(1 + \frac{0.712}{8}\frac{\lambda}{J_1^2} - \frac{1.699}{32}\frac{\lambda^2}{J_1^4} + \dots\right). \tag{4.62}$$

We will later also compare this result to other string solutions.

The special case of $\omega_1 = \omega_2$

If $\omega_1 = \omega_2$, then the equations of motion yield

$$\frac{\partial\gamma}{\partial\sigma} = 0 \qquad \Rightarrow \qquad \gamma(\sigma) = n\sigma \tag{4.63}$$

with an integer n, the winding number. The quantization is due to the closed string periodicity condition $\gamma(\sigma + 2\pi) = \gamma(\sigma) + 2\pi n$ for the 2π-periodic coordinate γ. The Virasoro constraints (4.35) lead to

$$\begin{aligned}
0 &= \dot{y}^m\dot{y}_m + \dot{x}^m\dot{x}_m + y'^m y'_m + x'^m x'_m \\
&= -\kappa^2 + \omega_1^2\cos^2\gamma + \omega_2^2\sin^2\gamma + \gamma'^2 \\
&= -\kappa^2 + \omega_1^2 + n^2 \\
\Rightarrow \quad \kappa &= \sqrt{n^2 + \omega_1^2}.
\end{aligned} \tag{4.64}$$

We can now evaluate the energy in terms of the other charges from equation (4.47),

$$J_1 = \sqrt{\lambda}\omega_1\int_0^{2\pi}\frac{d\sigma}{2\pi}\cos^2(n\sigma) = \frac{\sqrt{\lambda}\omega_1}{2},$$

$$J_2 = \sqrt{\lambda}\omega_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2(n\sigma) = \frac{\sqrt{\lambda}\omega_1}{2} = J_1$$

$$\Rightarrow \quad E = \sqrt{\lambda}\kappa = \sqrt{\lambda}\sqrt{n^2 + \omega_1^2} = \sqrt{\lambda}\sqrt{n^2 + \frac{4J_1^2}{\lambda}}$$

$$\stackrel{(4.64)}{\Rightarrow} \quad E(J_1) = 2J_1\sqrt{1 + \frac{\lambda n^2}{4J_1^2}}. \tag{4.65}$$

In the BMN limit with small λ/J^2, this is

$$E(J_1) = 2J_1\left(1 + \frac{\lambda n^2}{8J_1^2} - \frac{\lambda^2 n^4}{128 J_1^4} + \dots\right). \tag{4.66}$$

Comparing with the result for the folded string (4.62) shows that for $n = 1$, i.e. one winding, the energy of the circular string is bigger than the energy of the folded string in case both have the same charge J_1.

The circular string

If we have $q > 1$ in equation (4.45), then $d\gamma/d\sigma$ never vanishes. This means that the string is not folded but it is shaped like a circle. The steps are similar to the folded string, so we will only go through them briefly. First, we use the expression in equation (4.31) to find the angular momentum J_1. For this purpose, we use equation (4.2.3) to transform the integral

$$\begin{aligned}
J_1 &= \frac{\sqrt{\lambda}\omega_1}{2\pi} \int_0^{2\pi} d\gamma \frac{d\sigma}{d\gamma} \cos^2\gamma \\
&= \frac{\sqrt{\lambda}\omega_1}{2\pi} \int_0^{2\pi} d\gamma \frac{\cos^2\gamma}{\omega_{21}\sqrt{q - \sin^2\gamma}} \\
&= \frac{2\sqrt{\lambda}\omega_1}{\pi\omega_{21}} \int_0^{\pi/2} d\gamma \frac{1 - q + q - \sin^2\gamma}{\sqrt{q - \sin^2\gamma}} \\
&= \frac{2\sqrt{\lambda}\omega_1}{\pi\omega_{21}} \int_0^{\pi/2} d\gamma \left[\frac{1-q}{\sqrt{q}} \frac{1}{\sqrt{1 - q^{-1}\sin^2\gamma}} + \sqrt{q}\sqrt{1 - q^{-1}\sin^2\gamma}\right] \\
&= \frac{2\sqrt{\lambda}\omega_1}{\pi\omega_{21}} \left[\frac{1-q}{q}K(q^{-1}) + \sqrt{q}E(q^{-1})\right]. \tag{4.67}
\end{aligned}$$

In analogy with equation (4.53), we obtain

$$2\pi = \int_0^{2\pi} d\sigma = 4 \int_0^{\pi/2} \frac{d\gamma}{\omega_{21}\sqrt{q - \sin^2\gamma}} = \frac{4}{\omega_{21}\sqrt{q}}K(q^{-1}) \tag{4.68}$$

Performing the same steps as from equation (4.54) to equation (4.58), we finally obtain the relations

$$\frac{4\lambda}{\pi^2} = \frac{E^2}{K(q^{-1})^2} - \frac{J_2}{[(1-q)K(q^{-1}) + qE(q^{-1})]^2},$$
$$\frac{4q\lambda}{\pi^2} = \frac{J_2^2}{[K(q^{-1}) - E(q^{-1})]^2} - \frac{q^2 J_1^2}{[(1-q)K(q^{-1}) + qE(q^{-1})]^2}, \qquad (4.69)$$

which implicitly define the energy as a function of the two R-charges J_1 and J_2. If we expand q and E in powers of λ/J^2, $J = J_1 + J_2$, as in equation (4.59), we get an implicit expression for q_0,

$$\frac{J_2}{J} = q_0 \left(1 - \frac{E(q_0^{-1})}{K(q_0^{-1})} \right), \qquad (4.70)$$

as well as

$$E_0 = 1, \qquad E_1 = \frac{2}{\pi^2} E(q_0^{-1}) K(q_0^{-1}). \qquad (4.71)$$

This is a prediction for the energy up to one loop in the new coupling constant J^2/λ.

In chapter 5, we will see how we can reproduce these energies on the gauge theory side with the help of integrable spin chains. Therefore, these solutions provide a powerful test for the AdS/CFT correspondence. In this chapter, we have restricted ourselves to classical solutions. Frolov and Tseytlin have found a semi-classical quantization procedure for large R-charge strings [28, 30]. However, this quantization is beyond the scope of this dissertation. Consequently, we will focus purely on the planar limit on the gauge theory side, which corresponds to the free string limit here.

5 Spin chains and integrable systems

In this chapter, we explore how the gauge theory equivalents of the string theory solutions, we derived in the chapters 3 and 4, can be translated to spin chains and their spectra. The problem of finding scaling dimensions of operators will lead to the task of diagonalizing Hamiltonians of such spin chains, which was first realized by Minahan and Zarembo [53] as well as Beisert, Staudacher and collaborators [7, 6]. This is a quantum mechanical problem which has been considered for decades and such spin chains are known to be integrable, which Minahan and Zarembo first realized in the context of AdS/CFT [53]. We will find that the diagonalization problem will be solved by the so-called Bethe ansatz proposed in the 1930s [17]. Using this ansatz, we will be able to reproduce the predictions for the one-loop scaling dimensions of the gauge theory operators in the planar limit which we obtained from the string theory side by deriving energies of classical string solutions. A comprehensive review about spin chains and the Bethe ansatz in $\mathcal{N} = 4$ SYM was written by Minahan [54].

5.1 Gauge theory operators as spin chains

In chapter 4, we discussed strings stationary in AdS_5 and rotating in two orthogonal planes in the S^5 component of $AdS_5 \times S^5$. Their angular momenta in AdS_5 vanish, $S_1 = S_2 = 0$, but in the generic case they carry two non-vanishing R-charges J_1 and J_2. The third R-charge J_3 is also vanishing. In section 4.2.3, we managed to express the energy of string solutions with these charges as functions of the charges in a perturbative expansion with the BMN coupling $\lambda = \lambda/J^2$ as expansion parameter. On the gauge theory side, we should be able to reproduce the same functions as scaling dimensions of the corresponding operators in the BMN limit.

The R-symmetry rotates the six scalars ϕ_i, $i = 1, \ldots, 6$, among themselves. If we let J_1 be the generator of rotations in the $SO(2) \subset SO(6)$ subsector corresponding to the ϕ_1, ϕ_2-plane and J_2 the generator in the ϕ_3, ϕ_4-plane, then an operator with charges J_1 and J_2 is of the form

$$\mathcal{O}_\alpha^{J_1, J_2} = \text{Tr}\,[Z^{J_1} W^{J_2}] + \text{permutations} \tag{5.1}$$

where $Z = \phi_1 + i\phi_2$ and $W = \phi_3 + i\phi_4$. Operators of this form can be represented as an $s = 1/2$ spin chain. A factor of Z corresponds to spin down $|\downarrow\rangle$, a factor of W to spin up $|\uparrow\rangle$. Let us consider an example of an operator with $(J_1, J_2) = (5, 5)$:

$$\mathrm{Tr}\,[Z^2 W Z^2 W^3 Z W] = |\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\rangle_{\text{cyclic}} \tag{5.2}$$

where cyclicity implies that states, whose strings of spins are equal after some cyclic permutation, must be equivalent. Obviously, we encounter the problem of operator mixing since all the operators with J_1 factors of Z and J_2 factors of W will have the classical dimension $\Delta_0^{\mathcal{O}_\alpha^{J_1, J_2}} = J_1 + J_2$. As described in section 2.2.6, we need to diagonalize the action of the dilatation operator D on the operators that are classically degenerate,

$$D \circ \mathcal{O}_\alpha^{J_1, J_2} = -i\Delta_{\alpha\beta} \mathcal{O}_\beta^{J_1, J_2}, \tag{5.3}$$

i.e. we must diagonalize the matrix $\Delta_{\alpha\beta}$. If we calculate the dilatation operator in perturbation theory, we find [57]

$$D = \sum_{n=0}^{\infty} D^{(n)} \tag{5.4}$$

with

$$D^{(0)} = \mathrm{Tr}\,(Z\breve{Z} + W\breve{W}), \quad D^{(1)} = -\frac{g_{YM}^2}{8\pi^2} \mathrm{Tr}\,[Z, W][\breve{Z}, \breve{W}] \quad \text{where} \quad \breve{Z}_{ij} = \frac{\partial}{\partial Z_{ji}}. \tag{5.5}$$

Obviously, the classical component of D satisfies

$$D^{(0)} \mathcal{O}_\alpha^{J_1, J_2} = J_1 + J_2 \tag{5.6}$$

since, for example,

$$\begin{aligned}
\mathrm{Tr}\,(Z\breve{Z})\mathrm{Tr}\,[ZWZW] &= Z_{ij}\breve{Z}_{ji}(Z_{kl}W_{lm}Z_{mn}W_{nk}) \\
&= Z_{ij}(\delta_{ik}\delta_{jl}W_{lm}Z_{mn}W_{nk} + Z_{kl}W_{lm}\delta_{im}\delta_{jn}W_{nk}) \\
&= 2\mathrm{Tr}\,[ZWZW].
\end{aligned} \tag{5.7}$$

This result generalises to any number of Z's and holds analogously for W. It is what we expect from the classical part of the dilatation operator. Since the scalars have mass dimension one, the classical scaling dimension is just the sum of the mass dimensions of the involved fields.

We remember that we are interested in the BMN limit later, which is given by $J \to \infty$, $N \to \infty$, such that N/J^2 is constant. In this limit g_{YM} remains constant. Here, we will simplify further by additionally considering the planar limit which

corresponds to free strings on the $AdS_5 \times S^5$ side. We saw in section 3.6 that a natural effective genus counting parameter $g_2 = J^2/N$ arises in the BMN limit and a generic correlation function receives contributions from all orders in this genus expansion. This parameter remains tuneable after taking the limit. Therefore, in a generic case, a correlation function will have a double expansion in the parameters

$$\lambda' = \frac{g_{YM}^2 N}{J^2} \qquad \text{and} \qquad g_2 = \frac{J^2}{N} \tag{5.8}$$

where λ' controls quantum corrections and g_2 the genus expansion. If we take $g_2 \to 0$, i.e. go to the planar limit, in addition to the BMN limit, then this includes $g_{YM} \to 0$ to keep λ' constant. On the string theory side, this translates to $g_s \to 0$, i.e. the free string limit.

Now, we arrive at a crucial point. We will derive that, if we consider the planar limit, $D^{(1)}$ from equation (5.5) only acts on neighbouring fields in the string of fields that make up an operator. Consider $D^{(1)}$ acting on Z and W separated by two matrices $\mathcal{A}$ and $\mathcal{B}$, then

$$\begin{aligned} D^{(1)}\text{Tr}\,(Z\mathcal{A}W\mathcal{B}) &= \text{Tr}\,[Z,W][\check{Z},\check{W}]\,\text{Tr}\,(Z\mathcal{A}W\mathcal{B}) \\ &= [Z,W]_{ij}(\check{Z}_{jk}\check{W}_{ki} - \check{W}_{jk}\check{Z}_{ki})Z_{mn}\mathcal{A}_{np}W_{pq}\mathcal{B}_{qm} \\ &= [Z,W]_{ij}(\delta_{jn}\delta_{km}\delta_{kq}\delta_{ip} - \delta_{jq}\delta_{kp}\delta_{kn}\delta_{im})\mathcal{A}_{np}\mathcal{B}_{qm} \\ &= [Z,W]_{ij}(\mathcal{A}_{ji}\mathcal{B}_{kk} - \mathcal{A}_{kk}\mathcal{B}_{ji}) \\ &= -\text{Tr}\,(\mathcal{A})\,\text{Tr}\,([Z,W]\mathcal{B}) + \text{Tr}\,(\mathcal{B})\,\text{Tr}\,([Z,W]\mathcal{A}). \end{aligned} \tag{5.9}$$

Therefore, there is only a planar contribution if $\mathcal{A} = 1$ or $\mathcal{B} = 1$ as the contribution of a multi-trace operator is non-planar. From equation (5.9), we can conclude that the planar part of $D^{(1)}$, which is given in equation (5.5), consists of the identity minus a permutation when acting on neighbouring fields. It vanishes if the neighbouring fields are of the same type, Z or W. Furthermore, it does not act on non-neighbouring fields. Therefore, we can write

$$D^{(1)}_{\text{planar}} = \frac{\lambda}{8\pi^2} \sum_{i=1}^{L} (\mathbf{1}_{i,i+1} - P_{i,i+1}). \tag{5.10}$$

where L is the length of the string of operators. $P_{i,i+1}$ exchanges the fields in the ith and $(i+1)$th position in the string of operators and $P_{L,L+1} = P_{L,1}$. When comparing to equation (5.5), we see that there is an extra factor of N in the prefactor which comes from the trace of the unit matrix in equation (5.9). When studying this planar limit of the one-loop dilatation operator, Minahan and Zarembo recognized

this operator to be the Heisenberg $\mathrm{XXX}_{1/2}$ quantum spin chain Hamiltonian [53],

$$D_{\mathrm{planar}}^{(1)} = \frac{\lambda}{8\pi^2} \underbrace{\mathcal{H}_{\mathrm{XXX}_{1/2}}}_{\equiv Q_2} . \tag{5.11}$$

Consequently, the expression of an operator, which is a string of Z's and W's, in terms of a spin chain, which was introduced above, is not just a notational simplification but captures the fact that the action of a planar dilatation on this operator is equivalent to the action of the Hamiltonian on the corresponding spin chain. The diagonalization of equation (5.3) is therefore equivalent to finding the eigenstates and energies of a Heisenberg spin chain with J_1 spins down and J_2 spins up. The ground state is $|\downarrow\downarrow\ldots\downarrow\rangle_{\mathrm{cyclic}} \Leftrightarrow \mathrm{Tr}\left(Z^L\right)$ and there are J_2 excitations, called magnons in the spin chain picture. Solving this system is a well-studied, long-known problem. It turns out that the spin chain is integrable, i.e. there exists a set of $L-1$ charges Q_k which commute with the Hamiltonian and among themselves. The effect of these higher charges is that multi-body scattering processes factorize into two-body interactions. Therefore, it is sufficient to find the two-body scattering matrix to solve higher scattering processes, i.e. we can derive the M magnon case from studying the two magnon solution. In the next section, we will solve this two-magnon case using the Bethe ansatz [17]. A full treatment of the subject of integrability is beyond the scope of this dissertation, but there exists a number of reviews about integrability in the AdS/CFT correspondence [25, 63, 14]. Rather than describing integrability in depth, we will focus on its application in our dual gauge theory description of rotating strings.

5.2 The coordinate Bethe ansatz

The vacuum state for our theory of $s = 1/2$ spin chains is $|\downarrow\downarrow\ldots\downarrow\rangle$. Let us forget about the cyclicity for a moment and reintroduce it later. For now, the spin chain is only periodic. We denote an excited state by $|x_1, x_2, \ldots, x_M\rangle$, $x_1 < x_2 < \ldots < x_M$, $M < L$, where the x_i are the positions of the up-spins (magnons) in an M-magnon state. L still denotes the length of the spin chain. The Hamiltonian (5.10) does not change the number of magnons when acting on a state. This corresponds to the statement, that the dilatation operator does not mix states of different classical dimensions.

Now, we will try to find the M magnon energy eigenstates, i.e. we diagonalize the

Hamiltonian. For one magnon, this is done by a Fourier transformation

$$|\psi(p_1)\rangle \equiv \sum_{i=1}^{L} e^{ip_1 x} |x\rangle. \tag{5.12}$$

The reason is that

$$Q_2|x\rangle = |x\rangle - |x-1\rangle + |x\rangle - |x+1\rangle$$
$$= 2|x\rangle - |x-1\rangle - |x+1\rangle \tag{5.13}$$

where we recall the Hamiltonian Q_2 from equation (5.11). From this, we obtain

$$Q_2|\psi(p_1)\rangle = \sum_{x=1}^{L} e^{ip_1 x} \left(2|x\rangle - |x-1\rangle - |x+1\rangle\right)$$
$$= \sum_{x=1}^{L} \left(2e^{ip_1 x}|x\rangle - e^{ip_1} e^{ip_1(x-1)}|x-1\rangle - e^{-ip_1} e^{ip_1(x+1)}|x+1\rangle\right)$$
$$= \left(2 - e^{ip_1} - e^{-ip_1}\right)|\psi(p_1)\rangle$$
$$= 4\sin^2\left(\frac{p_1}{2}\right)|\psi(p_1)\rangle \tag{5.14}$$

where we have used periodicity of the spin chain to make the shifts $x \to x+1$ and $x \to x-1$. The periodic boundary conditions require $p_1 = 2\pi k/L$, $k \in \mathbb{Z}$. The spectrum of the spin chain is therefore given by the states

$$\sum_{x=1}^{L} e^{2\pi i k/L}|x\rangle, \qquad k \in \mathbb{Z}, \tag{5.15}$$

with corresponding energies $E_2 = 4\sin^2(\pi k/L)$.

The two magnon case is more complicated. We can write a general two magnon state as

$$|\psi(p_1, p_2)\rangle = \sum_{1 \le x_1 < x_2 \le L} \psi(x_1, x_2)|x_1, x_2\rangle. \tag{5.16}$$

For $x_2 > x_1 + 1$ we have

$$Q_2|x_1, x_2\rangle = 4|x_1, x_2\rangle - |x_1 - 1, x_2\rangle - |x_1 + 1, x_2\rangle - |x_1, x_2 - 1\rangle - |x_1, x_2 + 1\rangle \tag{5.17}$$

and for $x_2 = x_1 + 1$

$$Q_2|x_1, x_2\rangle = 2|x_1, x_2\rangle - |x_1 - 1, x_2\rangle - |x_1, x_2 + 1\rangle. \tag{5.18}$$

Therefore, from the Schrödinger equation $Q_2|\psi(p_1, p_2)\rangle = E_2|\psi(p_1, p_2)\rangle$, we get

$$E_2\psi(x_1, x_2) = 4\psi(x_1, x_2) - \psi(x_1 - 1, x_2) - \psi(x_1 + 1, x_2)$$
$$- \psi(x_1, x_2 - 1) - \psi(x_1, x_2 + 1) \tag{5.19}$$

for $x_2 > x_1 + 1$ and

$$E_2\psi(x_1, x_2) = 2\psi(x_1, x_2) - \psi(x_1 - 1, x_2) - \psi(x_1, x_2 + 1) \tag{5.20}$$

for $x_2 = x_1 + 1$. We employ Bethe's ansatz [17] to solve equation (5.19). This ansatz assumes $\psi(x_1, x_2)$ to be a superposition of an incoming plane wave and an outgoing, scattered wave,

$$\psi(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)} + S(p_2, p_1)e^{i(p_2 x_1 + p_1 x_2)} \tag{5.21}$$

where $S(p_2, p_1)$ is the scattering matrix. In the scattered part of the wave the two momenta are exchanged. Plugging this ansatz into equation (5.19), we find

$$\begin{aligned}
E_2\psi(x_1, x_2) &= 4\psi(x_1, x_2) - e^{-ip_1}e^{i(p_1 x_1 + p_2 x_2)} - e^{-ip_2}S(p_2, p_1)e^{i(p_2 x_1 + p_1 x_2)} \\
&\quad - e^{ip_1}e^{i(p_1 x_1 + p_2 x_2)} - e^{ip_2}S(p_2, p_1)e^{i(p_2 x_1 + p_1 x_2)} \\
&\quad - e^{-ip_2}e^{i(p_1 x_1 + p_2 x_2)} - e^{-ip_1}S(p_2, p_1)e^{i(p_2 x_1 + p_1 x_2)} \\
&\quad - e^{ip_2}e^{i(p_1 x_1 + p_2 x_2)} - e^{ip_1}S(p_2, p_1)e^{i(p_2 x_1 + p_1 x_2)} \\
&= \left(4 - e^{-ip_1} - e^{ip_1} - e^{-ip_2} - e^{ip_2}\right)\psi(x_1, x_2) \\
&= \left(4\sin^2\left(\frac{p_1}{2}\right) + 4\sin^2\left(\frac{p_2}{2}\right)\right)\psi(x_1, x_2) \\
\Rightarrow \quad E_2 &= 4\sin^2\left(\frac{p_1}{2}\right) + 4\sin^2\left(\frac{p_2}{2}\right). \tag{5.22}
\end{aligned}$$

We plug this solution into equation (5.20) and a tedious, but straightforward calculation shows that

$$S(p_1, p_2) = \frac{u_1 - u_2 + i}{u_1 - u_2 - i} \qquad \text{with} \qquad u_i = \frac{1}{2}\cot\left(\frac{p_i}{2}\right). \tag{5.23}$$

The u_i, $i = 1, 2$, are called Bethe roots. Furthermore, L-periodicity implies

$$\begin{aligned}
&\psi(x_1, x_2) = \psi(x_2, x_1 + L) \\
\Rightarrow \quad & e^{i(p_1 x_1 + p_2 x_2)} + S(p_2, p_1)e^{i(p_2 x_1 + p_1 x_2)} \\
&\qquad = e^{i(p_1 x_2 + p_2(x_1 + L))} + S(p_2, p_1)e^{i(p_2 x_2 + p_1(x_1 + L))} \\
\Rightarrow \quad & 1 = S(p_2, p_1)e^{ip_1 L} \quad \text{and} \quad S(p_2, p_1) = e^{ip_2 L} \\
\Rightarrow \quad & S(p_1, p_2) = e^{ip_1 L} \quad \text{and} \quad S(p_2, p_1) = e^{ip_2 L} \tag{5.24}
\end{aligned}$$

where we have used $S(p_2, p_1)^{-1} = S(p_1, p_2)$ and the linear independence of different plane waves. These are the so-called Bethe equations. By multiplying the two relations in equation (5.24) we find

$$1 = e^{i(p_1+p_2)L} \qquad \Rightarrow \qquad p_1 + p_2 = \frac{2\pi k}{L}, \quad k \in \mathbb{Z}. \tag{5.25}$$

As we already mentioned, the knowledge of the two-body problem is sufficient to solve the M-body problem in an integrable theory. This phenomenon is called factorized scattering. All higher scattering processes between more than two bodies factorize into two-body interactions. Two bodies interact via elastic scattering, i.e. exchange of their momenta, with the S-matrix $S(p_2, p_1)$. The M-magnon wave function is [17]

$$\psi(x_1, \ldots, x_M) = \sum_{\sigma \in S_M} \exp\left[i \sum_{i=1}^{M} p_{\sigma(i)} x_i + \frac{i}{2} \sum_{i<j} \theta_{\sigma(i)\sigma(j)} \right] \tag{5.26}$$

where the θ_{ij} are defined by

$$S(p_i, p_j) = \exp(i\theta_{ij}). \tag{5.27}$$

where $S(p_i, p_j)$ is given by equation (5.23) and consequently $\theta_{ji} = -\theta_{ij}$. The Bethe equations (5.24) become M equations,

$$e^{ip_k L} = \prod_{i=1, i \neq k}^{M} S(p_k, p_i) \qquad \Rightarrow \qquad \sum_{i=1}^{M} p_k = \frac{2\pi m}{L}, \quad m \in \mathbb{Z}, \tag{5.28}$$

which can be written in an alternative way using

$$\begin{aligned} e^{ip_k} &= \frac{e^{i\frac{p_k}{2}}}{e^{-i\frac{p_k}{2}}} = \frac{\cos\left(\frac{p_k}{2}\right) + i\sin\left(\frac{p_k}{2}\right)}{\cos\left(\frac{p_k}{2}\right) - i\sin\left(\frac{p_k}{2}\right)} \\ &= \frac{\cot\left(\frac{p_k}{2}\right) + i}{\cot\left(\frac{p_k}{2}\right) - i} = \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \end{aligned} \tag{5.29}$$

which yields

$$1 = e^{ip_k L} \prod_{i=1, i \neq k}^{M} S(p_i, p_k) = \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L \prod_{i=1, i \neq k}^{M} \frac{u_i - u_k + i}{u_i - u_k - i}. \tag{5.30}$$

The energies (5.22) are

$$E_2 = \sum_{i=1}^{M} 4\sin^2\left(\frac{p_i}{2}\right) \tag{5.31}$$

and they can be expressed in terms of the Bethe roots,

$$\frac{1}{u_i^2 + \frac{1}{4}} = \frac{4}{\cot^2\left(\frac{p_i}{2}\right) + 1} = \frac{4\sin^2\left(\frac{p_i}{2}\right)}{\cos^2\left(\frac{p_i}{2}\right) + \sin^2\left(\frac{p_i}{2}\right)}$$

$$= 4\sin^2\left(\frac{p_i}{2}\right)$$

$$\Rightarrow \quad E_2 = \sum_{i=1}^{M} \frac{1}{u_i^2 + \frac{1}{4}}. \tag{5.32}$$

Next, we want to reintroduce cyclicity of the spin chain. Therefore, we need to make the restriction equation (5.28) even more strict,

$$\sum_{i=1}^{M} p_i = 2\pi n, \quad n \in \mathbb{Z}. \tag{5.33}$$

The reason is that, if we shift the position of all magnons by one, cyclicity requires the wave function to remain the same, so

$$\psi(x_1 + 1, \ldots, x_M + 1) = \sum_{\sigma \in S_M} \exp\left[i\sum_{i=1}^{M} p_{\sigma(i)}(x_i + 1) + \frac{i}{2}\sum_{i<j}\theta_{\sigma(i)\sigma(j)}\right]$$

$$= \sum_{\sigma \in S_M} \exp\left[i\sum_{i=1}^{M} p_{\sigma(i)}x_i + i\sum_{i=1}^{M} p_{\sigma(i)} + \frac{i}{2}\sum_{i<j}\theta_{\sigma(i)\sigma(j)}\right]$$

$$= \exp\left[i\sum_{i=1}^{M} p_i\right]\sum_{\sigma \in S_M} \exp\left[i\sum_{i=1}^{M} p_{\sigma(i)}x_i + \frac{i}{2}\sum_{i<j}\theta_{\sigma(i)\sigma(j)}\right]$$

$$= \exp\left[i\sum_{i=1}^{M} p_i\right]\psi(x_1, \ldots, x_M) \tag{5.34}$$

where we have used that $\sum_{i=1}^{M} p_{\sigma(i)} = \sum_{i=1}^{M} p_i$ for any permutation $\sigma \in S_M$. Now, we can conclude that

$$\exp\left[i\sum_{i=1}^{M} p_i\right] = 1 \quad \Rightarrow \quad \sum_{i=1}^{M} p_i = 2\pi n, \quad n \in \mathbb{Z}. \tag{5.35}$$

This means the integer m in equation (5.28) has to be a multiple of L. We can express the constraint equation (5.33) in terms of the Bethe roots u_k, $k = 1, \ldots, M$, as

$$\prod_{i=1}^{M} \frac{u_i + \frac{i}{2}}{u_i - \frac{i}{2}} = 1 \tag{5.36}$$

where we used equation (5.29) to rewrite e^{ip_i} in terms of the Bethe roots.

Let us now diagonalize the two magnon case exactly. We can set $p = p_1 = -p_2$

because of equation (5.33) since the two momenta p and $p + 2\pi$ are equivalent. The Bethe equations 5.24 become

$$
\begin{aligned}
e^{ipL} = S(p, -p) &= \frac{\frac{1}{2}\cot\left(\frac{p}{2}\right) - \frac{1}{2}\cot\left(-\frac{p}{2}\right) + i}{\frac{1}{2}\cot\left(\frac{p}{2}\right) - \frac{1}{2}\cot\left(-\frac{p}{2}\right) - i} \\
&= \frac{\cot\left(\frac{p}{2}\right) + i}{\cot\left(\frac{p}{2}\right) - i} = e^{ip}
\end{aligned}
$$

$$
\Rightarrow \quad e^{ip(L-1)} = 1
$$

$$
\Rightarrow \quad p = \frac{2\pi n}{L - 1}, \quad n \in \mathbb{Z} \tag{5.37}
$$

Therefore, the energy eigenvalues (5.22) are

$$
E_2 = 8 \sin^2\left(\frac{\pi n}{L - 1}\right) \tag{5.38}
$$

which yields the scaling dimensions of the two magnon operators to be

$$
\Delta^{(1)}_{\text{planar}} = \frac{\lambda}{\pi^2} \sin^2\left(\frac{\pi n}{J + 1}\right) \tag{5.39}
$$

after reintroducing the prefactor $\lambda/8\pi^2$ from equation (5.11) and identifying the length of the spin chain L with the sum of the R-charges $J + 2 = J_1 + J_2$. The corresponding two magnon operators are [11, 53]

$$
\mathcal{O}^{(J,2)}_n = \sum_{p=0}^{J} \cos\left(\pi n \frac{2p + 1}{J + 1}\right) \text{Tr}\left(W Z^p W Z^{J-p}\right). \tag{5.40}
$$

In the BMN limit, $J \to \infty$, $N \to \infty$ and λ/J^2 fixed, we see that the scaling dimension in equation (5.39) obeys

$$
\Delta^{(1)}_{\text{planar}} = \frac{\lambda}{\pi^2} \sin^2\left(\frac{\pi n}{J + 1}\right) \quad \to \quad \frac{n^2 \lambda}{J^2}. \tag{5.41}
$$

Significantly, this agrees with the first term in the expansion of the level-two energy spectrum of the plane-wave superstring in equation (3.57),

$$
\frac{E_{\text{lc}}}{\mu} = 2\sqrt{1 + \frac{n^2 \lambda}{J^2}} \approx 2 + \frac{n^2 \lambda}{J^2} + \dots. \tag{5.42}
$$

We have seen that these level-two excitations correspond to operators of the form $\text{Tr}\left[\phi_i Z^l \phi_j Z^{J-l}\right]$. If we expand $W = \phi_3 + i\phi_4$, we see that the two magnon operators $\mathcal{O}^{(J,2)}_n$ from equation (5.40) are also of this form. Therefore, we had to recover this one-loop result.

5.3 The thermodynamic Bethe ansatz

In order to find the scaling dimensions of the operators corresponding to the spinning
folded and circular string solutions we found in section 4.2.3, we have to consider
spin chains with $L \to \infty$ as before. Unlike earlier, the number of magnons M is
also becoming infinite. This is necessary for J_2/J (here, $J = L$ and $J_2 = M$) not
to vanish in the BMN limit, where $J \to \infty$. Consequently, we keep J_2/J fixed in
the $J \to \infty$ limit. Such thermodynamic solutions were explored in [9, 8, 4, 24].
The description is mathematically non-trivial and requires some tools from complex
analysis. Hence, we will focus on the main ideas and refer to the given references
for the details.

To obtain the thermodynamic limit, we take the logarithm of the Bethe equations
(5.30),

$$L \ln \left(\frac{u_i + \frac{i}{2}}{u_i - \frac{i}{2}} \right) = \sum_{\substack{k=1 \\ k \neq i}}^{M} \ln \left(\frac{u_i - u_k + i}{u_i - u_k - i} \right) - 2\pi i n_i. \tag{5.43}$$

The n_i, $i = 1, \ldots, M$, are arbitrary integers that depend on the choice of a branch
of the complex logarithm. We consider the limit $L \to \infty$ in which the p_i scale as
$1/L$. Therefore, the u_i scale as L since $\cot(x) = 1/x + \mathcal{O}(x^0)$. We want to simplify
equation (5.43) in this thermodynamic limit. For this, we first observe that

$$\ln \left(\frac{1 + \frac{ix}{2}}{1 - \frac{ix}{2}} \right) = ix + \mathcal{O}(x^3). \tag{5.44}$$

Using this in equation (5.43) yields

$$\frac{1}{u_i} = \frac{2\pi n_i}{L} + \frac{2}{L} \sum_{\substack{k=1 \\ k \neq i}} \frac{1}{u_i - u_k}. \tag{5.45}$$

in the limit $u_i \sim L \to \infty$.

In the thermodynamic limit, the roots u_i are located on smooth contours C_n,
$C = \bigcup_n C_n$, in the complex plane, which allows us to introduce the Bethe root
density

$$\rho(u) = \frac{1}{M} \sum_{i=1}^{M} \delta(u - u_i) \qquad \Rightarrow \qquad \int_C du\, \rho(u) = 1. \tag{5.46}$$

Using this density, equation (5.43) can be written as

$$\int_C dv\, \frac{\rho(v)u}{v - u} = -\frac{1}{2\alpha} + \frac{\pi n_{C(u)} u}{\alpha} \qquad \text{with} \quad u \in C, \alpha = \frac{M}{L}. \tag{5.47}$$

The $n_{C(u)}$ are called mode numbers and they are constant on each smooth part of
the contour C_n. These numbers and the selection of components C_n determine the
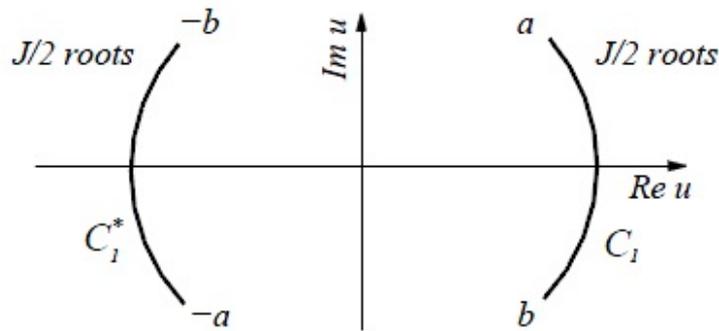
Figure 5.1: Two-component support $C = C_1 \cup C_1^*$ of the Bethe root density in the solution of the Bethe equations corresponding to the folded, rotating string solution from section 4.2.3. The image is taken from [56].

solutions to the Bethe equations (5.47). The energy of the solutions, which we have written in terms of the Bethe roots in equation (5.32), becomes

$$E_2 = M \int_C du \, \frac{\rho(u)}{u}. \tag{5.48}$$

Now, the task is to find the contours and mode numbers corresponding to our string solutions of a folded and a circular string rotating in the S^5 component of $AdS_5 \times S^5$ from section 4.2.3. It turns out that the folded string belongs to a two-component support of the root density, $C = C_1 \cup C_1^*$, see figure 5.1, with $n_{C_1} = -1$ and $n_{C_1^*} = 1$. Indeed, a very complicated calculation shows that this choice reproduces the one-loop energy from the string theory calculation, which we have performed in section 4.2.3 with the results in the equations (4.60) and (4.61) [56].

The analysis of the circular string solution is similar. The support of the Bethe root density is located on the imaginary axis and is symmetric, $\rho(u) = \rho(-u)$. The result is again in perfect agreement with the string theory results from equations (4.70) and (4.71) [56].

We conclude that we have found a powerful test of the AdS/CFT correspondence. The large R-charge sector is perturbatively accessible from both sides of the duality. Our calculations of the one-loop corrections to the energy on the string theory side and the scaling dimension on the gauge theory side agree for associated pairs of string solutions as well as corresponding operators.

6 Conclusions and outlook

In this dissertation, we studied how we can test the non-BPS sector of the AdS/CFT correspondence by comparing string spectra to the scaling dimensions of the dual gauge theory operators. We derived that a way to circumvent the problems, that arise from the fact that AdS/CFT is a strong/weak duality, is to study states and corresponding operators with large charges. Several examples of such highly charged classical solutions of the string σ-model in $AdS_5 \times S^5$ have been presented. We have seen that it is possible to identify the gauge theory equivalents of highly charged string theory states. Due to the large charges, these gauge theory operators contain a big number of fields which leads to the problem of operator mixing because the more fields an operator contains, the higher the classical degeneracy. When the involved charges are angular momenta in S^5, the task of finding the eigenstates of the dilatation operator can be translated to the problem of diagonalizing the Hamiltonian of spin chains. We have studied this technique to one loop order in the planar limit and found an integrable spin chain model. This model can be solved using the Bethe ansatz to find the eigenstates of the Hamiltonian and thus the scaling dimensions of operators in the gauge theory. As conjectured by the AdS/CFT duality, these scaling dimensions agree with the energies of the dual string states.

In their original paper, that introduced the integrable spin chain structure, Minahan and Zarembo considered the full scalar sector [53], which leads to an $SO(6)$ spin chain of which we have only covered an $SU(2)$ subsector. From the scalar sector we can generalize to the full theory obtaining a $PSU(2,2|4)$ spin chain, which is integrable at one loop and was studied by Beisert and Staudacher [13]. This spin chain has an infinite number of degrees of freedom at every site and its thermodynamic limit leads to the subject of spectral curves [10].

A natural question would be if integrable models also appear at higher loop orders or if it is just an artefact of the one loop approximation. Higher loop corrections of the dilatation operator have been derived corresponding to spin chains with interactions of an increasing range, which have been reviewed in [61]. The two loop correction in the $SU(2)$ subsector has been shown to be integrable and higher corrections have been found conjecturing integrability [6]. Higher rank chains have firstly been studied in [12] and two and three loop corrections in the $SU(2|3)$ subsector

have been proven to be integrable. There has been progress in other subsectors as well and for those we refer to the review [61] and the references within it.

We should also be able to find integrable structures on the string theory side of the duality. Integrability of the gauge theory in the planar limit hints at a hidden integrable structure in the $AdS_5 \times S^5$ string sigma model and its classical integrability has in fact been shown in [15].

Another question is if integrability can also be found in the non-planar sector which is dual to interacting strings. A review about aspects of non-planarity is due to Kristjansen [46]. Attempts to diagonalize the non-planar version of the dilatation operator have shown no signs of integrability so far. There are still attempts to find traces of integrable structures in n point functions or Wilson loops [46].

Finally, we want to mention one more possible direction for further research. It would be interesting to study if integrability extends to more phenomenological gauge theories such as QCD. It would be exciting to find hidden symmetries that are not obvious from the Lagrangian but only reveal themselves in the quantum theory. There is, in fact, a growing amount of evidence for such hidden symmetries and the progress in this field has been summarized in [44].

List of Figures

Bibliography

[1]   L. Andrianopoli and S. Ferrara. "K-K excitations on AdS(5) x S^5 as N=4 'primary' superfields". *Phys.Lett.* B430 (1998), pp. 248–253. arXiv:`hep-th/9803171` [`hep-th`].

[2]   L. Andrianopoli and S. Ferrara. "'Nonchiral' primary superfields in the AdS_{d+1} / CFT$_d$ correspondence". *Lett.Math.Phys.* 46 (1998), pp. 265–279. arXiv:`hep-th/9807150` [`hep-th`].

[3]   G. Arutyunov et al. "Exceptional nonrenormalization properties and OPE analysis of chiral four point functions in N=4 SYM(4)". *Nucl.Phys.* B620 (2002), pp. 380–404. arXiv:`hep-th/0103230` [`hep-th`].

[4]   G. Arutyunov and M. Staudacher. "Matching higher conserved charges for strings and spins". *JHEP* 0403 (2004), p. 004. arXiv:`hep-th/0310182` [`hep-th`].

[5]   C. A. Bayona and N. R. Braga. "Anti-de Sitter boundary in Poincare coordinates". *Gen.Rel.Grav.* 39 (2007), pp. 1367–1379. arXiv:`hep-th/0512182` [`hep-th`].

[6]   N. Beisert, C. Kristjansen, and M. Staudacher. "The Dilatation operator of conformal N=4 superYang-Mills theory". *Nucl.Phys.* B664 (2003), pp. 131–184. arXiv:`hep-th/0303060` [`hep-th`].

[7]   N. Beisert et al. "BMN gauge theory as a quantum mechanical system". *Phys.Lett.* B558 (2003), pp. 229–237. arXiv:`hep-th/0212269` [`hep-th`].

[8]   N. Beisert et al. "Precision spectroscopy of AdS / CFT". *JHEP* 0310 (2003), p. 037. arXiv:`hep-th/0308117` [`hep-th`].

[9]   N. Beisert et al. "Stringing spins and spinning strings". *JHEP* 0309 (2003), p. 010. arXiv:`hep-th/0306139` [`hep-th`].

[10] N. Beisert et al. "Complete spectrum of long operators in N=4 SYM at one loop". *JHEP* 0507 (2005), p. 030. arXiv:`hep-th/0503200` [`hep-th`].

[11] N. Beisert. "BMN operators and superconformal symmetry". *Nucl.Phys.* B659 (2003), pp. 79–118. arXiv:`hep-th/0211032` [`hep-th`].

[12] N. Beisert. "The su(2|3) dynamic spin chain". *Nucl.Phys.* B682 (2004), pp. 487–520. arXiv:`hep-th/0310252` [`hep-th`].

[13]   N. Beisert and M. Staudacher. "The N=4 SYM integrable super spin chain".
        Nucl.Phys. B670 (2003), pp. 439–463. arXiv:`hep-th/0307042` `[hep-th]`.

[14]   N. Beisert et al. "Review of AdS/CFT Integrability: An Overview". *Lett.Math.
        Phys.* 99 (2012), pp. 3–32. arXiv:`1012.3982` `[hep-th]`.

[15]   I. Bena, J. Polchinski, and R. Roiban. "Hidden symmetries of the AdS(5) x
        S^5 superstring". *Phys.Rev.* D69 (2004), p. 046002. arXiv:`hep-th/0305116`
        `[hep-th]`.

[16]   D. E. Berenstein, J. M. Maldacena, and H. S. Nastase. "Strings in flat space
        and pp waves from N=4 Super Yang-Mills". *JHEP* 0204 (2002). arXiv:`hep-
        th/0202021` `[hep-th]`.

[17]   H. Bethe. "On the theory of metals. 1. Eigenvalues and eigenfunctions for the
        linear atomic chain". *Z.Phys* 71 (1931), pp. 205–226.

[18]   M. Bianchi et al. "Anomalous dimensions in N=4 SYM theory at order g^4".
        Nucl.Phys. B584 (2000), pp. 216–232. arXiv:`hep-th/0003203` `[hep-th]`.

[19]   M. Blau et al. "A New maximally supersymmetric background of IIB super-
        string theory". *JHEP* 0201 (2002), p. 047. arXiv:`hep-th/0110242` `[hep-th]`.

[20]   L. Brink, O. Lindgren, and B. E. Nilsson. "The ultra-violet finiteness of the
        N=4 Yang-Mills theory". *Physics Letters B* 123.5 (1983), pp. 323 –328. URL:
        `www.sciencedirect.com/science/article/pii/0370269383912108`.

[21]   L. Brink, J. H. Schwarz, and J. Scherk. "Supersymmetric Yang-Mills the-
        ories". *Nuclear Physics B* 121.1 (1977), pp. 77 –92. ISSN: 0550-3213. URL:
        `www.sciencedirect.com/science/article/pii/0550321377903285`.

[22]   N. R. Constable et al. "PP wave string interactions from perturbative Yang-
        Mills theory". *JHEP* 0207 (2002), p. 017. arXiv:`hep-th/0205089` `[hep-th]`.

[23]   E. D'Hoker and D. Z. Freedman. "Supersymmetric gauge theories and the
        AdS/CFT correspondence" (2002). arXiv:`hep-th/0201253` `[hep-th]`.

[24]   J. Engquist, J. Minahan, and K. Zarembo. "Yang-Mills duals for semiclassical
        strings on AdS(5)xS(5)". *JHEP* 0311 (2003), p. 063. arXiv:`hep-th/0310188`
        `[hep-th]`.

[25]   L. Faddeev. "How algebraic Bethe ansatz works for integrable model" (1996),
        pp. 149–219. arXiv:`hep-th/9605187` `[hep-th]`.

[26]   S. Ferrara, C. Fronsdal, and A. Zaffaroni. "On N=8 supergravity on AdS(5)
        and N=4 superconformal Yang-Mills theory". *Nucl.Phys.* B532 (1998), pp. 153–
        162. arXiv:`hep-th/9802203` `[hep-th]`.

[27]   P. Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory.* Graduate
        Texts in Contemporary Physics. Springer, 1997.

[28] S. Frolov and A. A. Tseytlin. "Semiclassical quantization of rotating super-string in AdS(5) x S^5". *JHEP* 0206 (2002), p. 007. arXiv:`hep-th/0204226`.

[29] S. Frolov and A. A. Tseytlin. "Multispin string solutions in AdS(5) x S^5". *Nucl.Phys.* B668 (2003), pp. 77–110. arXiv:`hep-th/0304255 [hep-th]`.

[30] S. Frolov and A. A. Tseytlin. "Quantizing three spin string solution in AdS(5) x S^5". *JHEP* 0307 (2003), p. 016. arXiv:`hep-th/0306130 [hep-th]`.

[31] S. Frolov and A. A. Tseytlin. "Rotating string solutions: AdS/CFT duality in nonsupersymmetric sectors". *Phys.Lett.* B570 (2003), pp. 96–104. arXiv:`hep-th/0306143`.

[32] P. H. Ginsparg. "Applied Conformal Field Theory" (1988). arXiv:`hep-th/9108028 [hep-th]`.

[33] P. H. Ginsparg and G. W. Moore. "Lectures on 2-D gravity and 2-D string theory" (1993). arXiv:`hep-th/9304011 [hep-th]`.

[34] M. B. Green, J. H. Schwarz, and E. Witten. *Superstring theory.* Vol. I & II. Cambridge: University of Cambridge, 2002.

[35] D. J. Gross, A. Mikhailov, and R. Roiban. "Operators with large R charge in N=4 Yang-Mills theory". *Annals Phys.* 301 (2002), pp. 31–52. arXiv:`hep-th/0205066 [hep-th]`.

[36] D. J. Gross and F. Wilczek. "Asymptotically free gauge theories. 1". *Phys.Rev.* D8 (1973), pp. 3633–3652.

[37] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov. "A Semiclassical limit of the gauge/string correspondence". *Nucl.Phys.* B636 (2002), pp. 99–114. arXiv:`hep-th/0204051 [hep-th]`.

[38] S. Gubser, I. R. Klebanov, and A. M. Polyakov. "Gauge theory correlators from noncritical string theory". *Phys.Lett.* B428 (1998), pp. 105–114. arXiv:`hep-th/9802109 [hep-th]`.

[39] R. Haag, J. T. Lopuszanski, and M. Sohnius. "All Possible Generators of Supersymmetries of the S-Matrix". *Nucl.Phys.* B88 (1975), p. 257. DOI: `10.1016/0550-3213(75)90279-5`.

[40] G. 't Hooft. "A planar diagram theory for strong interactions". *Nucl.Phys.* B72 (1974), pp. 461–473. DOI: `10.1016/0550-3213(74)90154-0`.

[41] G. 't Hooft. "Dimensional reduction in quantum gravity" (1993). arXiv:`gr-qc/9310026 [gr-qc]`.

[42] Y. S. Kim and E. P. Wigner. "Space-time geometry of relativistic particles". *J.Math.Phys.* 31 (1990), pp. 55–60.

[43] G. Korchemsky. "Asymptotics of the Altarelli-Parisi-Lipatov evolution kernels of parton distributions". *Modern Physics Letters A* 04.13 (1989), pp. 1257–1276. eprint: `http://www.worldscientific.com/doi/pdf/10.1142/S0217732389001453`.

[44] G. Korchemsky. "Review of AdS/CFT Integrability, Chapter IV.4: Integrability in QCD and N=4 SYM". *Lett.Math.Phys.* 99 (2012), pp. 425–453. arXiv:`1012.4000 [hep-th]`.

[45] G. Korchemsky and G. Marchesini. "Structure function for large x and renormalization of Wilson loop". *Nucl.Phys.* B406 (1993), pp. 225–258. arXiv:`hep-ph/9210281 [hep-ph]`.

[46] C. Kristjansen. "Review of AdS/CFT Integrability, Chapter IV.1: Aspects of Non-Planarity". *Lett.Math.Phys.* 99 (2012), pp. 349–374. arXiv:`1012.3997 [hep-th]`.

[47] C. Kristjansen et al. "A New double scaling limit of N=4 super Yang-Mills theory and PP wave strings". *Nucl.Phys.* B643 (2002), pp. 3–30. arXiv:`hep-th/0205033 [hep-th]`.

[48] J. Maldacena. "The Gauge/gravity duality" (2011). arXiv:`1106.6073 [hep-th]`.

[49] J. M. Maldacena. "The Large N limit of superconformal field theories and supergravity". *Adv.Theor.Math.Phys.* 2 (1998), pp. 231–252. arXiv:`hep-th/9711200 [hep-th]`.

[50] S. Mandelstam. "Light Cone Superspace and the Ultraviolet Finiteness of the $\mathcal{N} = 4$ Model". *Nucl.Phys.* B213 (1983), pp. 149–168.

[51] R. Metsaev. "Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background". *Nucl.Phys.* B625 (2002), pp. 70–96. arXiv:`hep-th/0112044 [hep-th]`.

[52] R. Metsaev and A. A. Tseytlin. "Exactly solvable model of superstring in Ramond-Ramond plane wave background". *Phys.Rev.* D65 (2002), p. 126004. arXiv:`hep-th/0202109 [hep-th]`.

[53] J. Minahan and K. Zarembo. "The Bethe ansatz for N=4 superYang-Mills". *JHEP* 0303 (2003), p. 013. arXiv:`hep-th/0212208 [hep-th]`.

[54] J. A. Minahan. "Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in N=4 Super Yang-Mills". *Lett.Math.Phys.* 99 (2012), pp. 33–58. arXiv:`1012.3983 [hep-th]`.

[55] H. Nastase. "Introduction to AdS-CFT" (2007). arXiv:`0712.0689 [hep-th]`.

[56] J. Plefka. "Spinning strings and integrable spin chains in the AdS/CFT correspondence". *Living Rev.Rel.* 8 (2005), p. 9. arXiv:`hep-th/0507136 [hep-th]`.

[57] J. C. Plefka. "Lectures on the plane wave string/gauge theory duality". *Fortsch. Phys.* 52 (2004), pp. 264–301. arXiv:`hep-th/0307101` [`hep-th`].

[58] J. Polchinski. "Dirichlet Branes and Ramond-Ramond charges". *Phys.Rev.Lett.* 75 (1995), pp. 4724–4727. arXiv:`hep-th/9510017` [`hep-th`].

[59] J. Polchinski. "TASI lectures on D-branes" (1996), pp. 293–356. arXiv:`hep-th/9611050` [`hep-th`].

[60] B. Ralph and E. Plauschinn. *Introduction to Conformal Field Theory: With Applications to String Theory.* Lecture Notes in Physics. Berlin, Heidelberg: Springer, 2009.

[61] A. Rej. "Review of AdS/CFT Integrability, Chapter I.3: Long-range spin chains". *Lett.Math.Phys.* 99 (2012), pp. 85–102. arXiv:`1012.3985` [`hep-th`].

[62] A. Santambrogio and D. Zanon. "Exact anomalous dimensions of N=4 Yang-Mills operators with large R charge". *Phys.Lett.* B545 (2002), pp. 425–429. arXiv:`hep-th/0206079` [`hep-th`].

[63] D. Serban. "Integrability and the AdS/CFT correspondence". *J.Phys.* A44 (2011), p. 124001. arXiv:`1003.4214` [`hep-th`].

[64] M. F. Sohnius and P. C. West. "Conformal Invariance in N=4 Supersymmetric Yang-Mills Theory". *Phys.Lett.* B100 (1981), p. 245. DOI: `10.1016/0370-2693(81)90326-9`.

[65] K. Stelle. "Lectures on supergravity p-branes" (1996). arXiv:`hep-th/9701088` [`hep-th`].

[66] A. Strominger and C. Vafa. "Microscopic origin of the Bekenstein-Hawking entropy". *Phys.Lett.* B379 (1996), pp. 99–104. arXiv:`hep-th/9601029` [`hep-th`].

[67] L. Susskind. "The World as a hologram". *J.Math.Phys.* 36 (1995), pp. 6377–6396. arXiv:`hep-th/9409089` [`hep-th`].

[68] D. Tong. *String Theory.* `damtp.cam.ac.uk/user/tong/string/string.pdf`. 2009.

[69] A. Tseytlin. "Review of AdS/CFT Integrability, Chapter II.1: Classical AdS5 x S5 string solutions". *Lett.Math.Phys.* 99 (2012), pp. 103–125. arXiv:`1012.3986` [`hep-th`].

[70] E. Witten. "Anti-de Sitter space and holography". *Adv.Theor.Math.Phys.* 2 (1998), pp. 253–291. arXiv:`hep-th/9802150` [`hep-th`].